

Valori și Vectori proprii

① Se consideră matricea $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Det. val. proprii de numeră și vect proprii corresp.

$$\begin{vmatrix} 4-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (4-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda) = 0$$

$$\Leftrightarrow (1-\lambda)(\cancel{4} - 6\lambda + \lambda^2) = 0$$

$$\Delta = 36 - 24 = 8$$

$$\Rightarrow \lambda_1 = 1$$

$$\lambda_{2,3} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}$$

$$\lambda_{\max} = 3 + \sqrt{2}$$

$$A \cdot \vec{x} = \lambda_{\max} \cdot \vec{x}$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (3 + \sqrt{2}) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 4x_1 + x_2 \\ x_1 + 2x_2 + x_3 \\ x_3 \end{pmatrix} = (3 + \sqrt{2}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \vec{x}_3 = 0$$

$$\begin{cases} 4x_1 + x_2 = (3 + \sqrt{2})x_1 \\ x_1 + 2x_2 = (3 + \sqrt{2})x_2 \end{cases} \Rightarrow \text{~~Arădă~~$$

$$\begin{cases} (1+\sqrt{2})x_1 + 2x_2 = 0 \\ x_1 + (1+\sqrt{2})x_2 = 0 \end{cases} \quad | \cdot (1-\sqrt{2}) \quad / +$$

$$(2-2\sqrt{2}-1-\sqrt{2})x_2 = 0$$

$$(1-3\sqrt{2})x_2 = 0 \quad ??$$

$$x \in \lambda_0 \} (\sqrt{2}+1, 1, 0) \}$$

②

$$A = \begin{pmatrix} n & n & 0 & \dots & 0 \\ 0 & n-1 & n & \dots & 0 \\ 0 & 0 & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & n \\ 0 & 0 & \dots & \dots & 1 \end{pmatrix}$$

$$a) \quad x(A) = ?$$

$$b) \quad k(\lambda_k) = \frac{\|x_k\|_\infty \cdot \|y_k\|}{|y_k^T \cdot x_k|}$$

$$\begin{matrix} x_k, \lambda_k & A \\ y_k, x_k & A^T \end{matrix}$$

$$p(\lambda) = \prod (\lambda - k) \Rightarrow \lambda_k = k \quad k=1:n$$

$$A \cdot x_k = \lambda_k \cdot x_k \Rightarrow (A - \lambda_k I_n) x_k = 0$$

$$\begin{pmatrix} n-k & n & 0 & \dots & 0 \\ 0 & n-1-k & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1-k \end{pmatrix} \begin{pmatrix} x_{k1} \\ \vdots \\ x_{kn} \end{pmatrix} = 0_n$$

$$\Rightarrow \begin{cases} (n-k)x_{k1} + nx_{k2} = 0 \\ (n-1-k)x_{k2} + nx_{k3} = 0 \\ \vdots \\ (2-k)x_{k,n-1} + nx_{kn} = 0 \\ (1-k)x_{kn} = 0 \end{cases}$$

$\Downarrow$

$$x_{kj} = 0, \quad j = k+1:n$$

$$(n-k)x_{k1} = -nx_{k2}$$

$$(n-k-1)x_{k2} = -nx_{k3}$$

$$x_{k,k-1} = -nx_{kk}$$

$$x_{k1} = \frac{(-n)^{k-1}}{(n-k)!} x_{kk}$$

$$x_{kj} = \frac{(-n)^{k-j}}{(n-k-j)!} \cdot x_{kk}$$

$$x_k = \begin{pmatrix} \frac{(-n)^{k-1}}{(n-k)!} \\ \vdots \\ +n \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot x_{kk}$$

$$y_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ n \\ \vdots \\ \frac{(-n)^{k-1}}{(n-k)!} \end{pmatrix} \cdot y_{kk}$$

$$K(x_k) = \frac{\left(\frac{n^{k-1}}{(n-k)!} \right)^2 \cdot x_{kk} \cdot y_{kk}}{x_{kk} \cdot y_{kk}}$$

$$\textcircled{3} \quad A \in \mathbb{R}^{n \times n}, \quad \lambda(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

$$a) \quad \lambda(A) = \lambda(A^T)$$

$$b) \quad \begin{array}{c} x_i, \lambda_i, A \\ y_j, \lambda_j, A^T \end{array} \quad \Bigg| \quad x_i \perp y_j, \quad i \neq j$$

$$c) \quad B = A - \lambda_i \frac{x_i y_i^T}{y_i^T x_i}, \quad \lambda(B), x(B)$$

Bew:

$$a) \quad P(A) = \det(A - \lambda I_n)$$

$$\begin{aligned} P(A^T) &= \det(A^T - \lambda I_n) = \det(A - \lambda I_n)^T = \det(A - \lambda I_n) = \\ &= P(A) \quad \Rightarrow \quad \lambda(A) = \lambda(A^T) \end{aligned}$$

$$b) \quad \langle x_i, y_j \rangle \stackrel{?}{=} 0$$

$$\begin{cases} Ax_i = \lambda_i x_i \\ A^T y_j = \lambda_j y_j \end{cases} \Rightarrow \begin{cases} Ax_i = \lambda_i x_i & | \cdot y_j^T \text{ b. s.} \\ y_j^T A = \lambda_j y_j^T & | \cdot x_i \text{ b. d.} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y_j^T A x_i = \lambda_i y_j^T x_i \\ y_j^T A x_i = \lambda_j y_j^T x_i \end{cases}$$

$$0 = (\lambda_i - \lambda_j) y_j^T x_i \quad \Rightarrow \quad y_j^T x_i = 0 \quad \Rightarrow$$

$$i \neq j \Rightarrow \lambda_i \neq \lambda_j$$

$$\Rightarrow \langle x_i, y_j \rangle = 0 \quad \Rightarrow \quad x_i \perp y_j$$

$$c) \det(B - \beta I_n) = \det\left(A - \lambda_i \cdot \frac{x_i y_i^T}{y_i^T x_i} - \beta I_n\right) = 0$$

$$B \cdot x_j = A \cdot x_j - \lambda_i \cdot \frac{x_i y_i^T}{y_i^T x_i} \cdot x_j =$$

← unu dintre vectorii proprii ai matricei A

$$= \begin{cases} 0 = \lambda_i \cdot 0 & , i=j \\ A x_j = \lambda_i x_j & , i \neq j \end{cases} \Rightarrow$$

$$\Rightarrow \lambda(B) = \{\lambda_1, \lambda_2, \dots, \lambda_i, 0, \lambda_{i+1}, \dots, \lambda_n\}$$

$$\chi(B) = \chi(A)$$

④ $A, B \in \mathbb{R}^{n \times n}$ asemenea,

$\exists T$ invers a.e. $B = T^{-1} \cdot A \cdot T$

$$a) \det(B) = \det(A)$$

$$b) \operatorname{Tr}(B) = \operatorname{Tr}(A)$$

$$\begin{aligned} a) \det(B) &= \det(T^{-1} \cdot A \cdot T) = \det(T^{-1}) \cdot \det(A) \cdot \det(T) = \\ &= \frac{1}{\det T} \cdot \det A \cdot \det T = \det A \end{aligned}$$

$$P_A = \det(A - \lambda I_n)$$

$$P_B = \det(B - \lambda I_n) = \det I_n \cdot \det(B - \lambda I_n) = \\ = \det(T^{-1} B T - \lambda T^{-1} T) = \det(A - \lambda I_n)$$

$$\textcircled{5} \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix} = A$$

$$P_0(\lambda) = 1, \quad p_1(\lambda) = \lambda - a_{11}, \quad p_k(\lambda) = (\lambda - a_{kk}) \cdot p_{k-1}(\lambda) - a_{k,k-1} \cdot a_{k-1,k} \cdot p_{k-2}(\lambda)$$

$$\text{Analogic, c\^o} \quad p_A(\lambda) = p_n(\lambda) = (\lambda - a_{nn}) p_{n-1}(\lambda) - a_{n,n-1} \cdot a_{n-1,n} \cdot p_{n-2}(\lambda)$$

Demo:

$$P_A(\lambda) = \det(A - \lambda I_n) = \\ = (\lambda - a_{nn}) \cdot \begin{vmatrix} a_{11} - \lambda & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} - \lambda \end{vmatrix} - a_{n,n-1} \cdot \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & 0 \\ a_{21} & a_{22} - \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-2,n-2} - \lambda \end{vmatrix}$$

$$= (\lambda - a_{nn}) \cdot p_{n-1}(\lambda) - a_{n,n-1} \cdot a_{n-1,n} \cdot \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & 0 \\ a_{21} & a_{22} - \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-2,n-2} - \lambda \end{vmatrix} =$$

$$= (\lambda - a_{nn}) \cdot p_{n-1}(\lambda) - a_{n,n-1} \cdot a_{n-1,n} \cdot p_{n-2}(\lambda)$$

Seu por induc\c{a}o