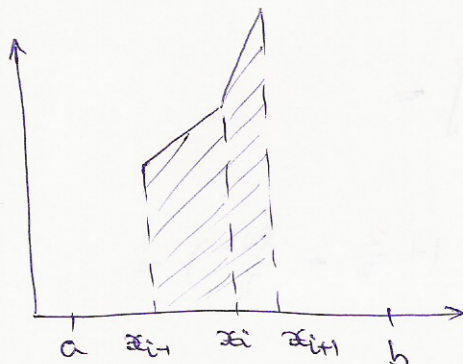
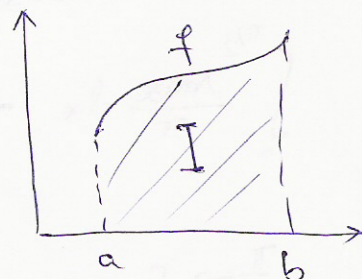


Integrarea numerică

$$I = \int_a^b f(x) dx = F(b) - F(a)$$



$$x_0 = a$$

$$x_i = a + ih, \quad i = 0, \dots, n$$

$$h = \frac{b-a}{n}$$

$$\begin{aligned} A &= \frac{(f(x_i) + f(x_{i-1})) \cdot h}{2} + \frac{(f(x_i) + f(x_{i+1})) \cdot h}{2} = \\ &= \frac{h}{2} (f(x_{i-1}) + f(x_{i+1}) + 2f(x_i)) \end{aligned}$$

$$\boxed{\int_a^b f(x) dx \approx \frac{h}{2} \left[f(a) + f(b) + 2 \sum_{i=1}^n f(x_i) \right]}, \quad h = \frac{b-a}{n}$$

Newton

$x_i = a + ih$

$$\boxed{\int_a^b f(x) dx \approx \frac{h}{3} \left[f(a) + f(b) + 4 \sum_{i=1}^n f(x_{2i-1}) + 2 \sum_{i=1}^n f(x_{2i}) \right]}$$

Simpson

$$h = \frac{b-a}{2n}$$

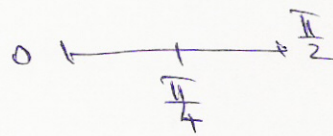
$$x_i = a + ih$$

$$i = 0, \dots, 2n$$

$$\int_a^b f(x) dx = \sum_{i=0}^n \Delta x_i f(x_i) + R$$

① Să se calculeze folosind met. Simpson cu 3 pc integr.

$$\int_0^{\pi/2} \frac{\tan x}{x} dx = I$$



$$I \approx \frac{\frac{\pi}{2} - 0}{6} \left[f(0) + f\left(\frac{\pi}{2}\right) + 4f\left(\frac{\pi}{4}\right) \right] =$$

$$= \frac{\pi}{12} \left(1 + \frac{2}{\pi} + 4 \cdot \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} \right) = \frac{\pi}{12} \left(1 + \frac{2}{\pi} + \frac{8\sqrt{2}}{\pi} \right) =$$

$$= \frac{2 + 8\sqrt{2} + \pi}{12}$$

sau ptr $\int_1^3 (\ln x^2 + 1) dx$

② a) Fie funcția f def pe $[a, b] \times [c, d]$. Să se def o form. de integrare cu 3 pc folosind formula lui Simpson ptr.

$$I = \int_a^b \int_c^d f(x, y) dx dy$$

b) $f: [1, 2] \times [1, 2] \rightarrow \mathbb{R}$ $f(x, y) = \frac{1}{x} + \frac{1}{y}$. Să se calculeze integr. prin met. exactă și prin met. aprox de la pc a)

$$\begin{aligned}
 I &= \int_a^b \int_c^d f(x,y) dx dy = \int_a^b dx \int_c^d f(x,y) dy = \\
 &= \int_a^b \left[\frac{d-c}{6} \left(f(x,c) + f(x,d) + 4f\left(x, \frac{d+c}{2}\right) \right) \right] dx = \\
 &= \frac{b-a}{6} \cdot \frac{d-c}{6} \left[f(a,c) + f(b,c) + 4f\left(\frac{a+b}{2}, c\right) + f(a,d) + f(b,d) + \right. \\
 &\quad \left. + 4f\left(\frac{a+b}{2}, d\right) + 4f\left(a, \frac{d+c}{2}\right) + 4f\left(b, \frac{d+c}{2}\right) + 16f\left(\frac{a+b}{2}, \frac{d+c}{2}\right) \right]
 \end{aligned}$$

$$I = \int_1^2 \int_1^2 \left(\frac{1}{x} + \frac{1}{y} \right) dx dy = 2 \ln 2 \approx 1,38$$

$$I \approx \frac{25}{18} \approx 1,38$$

Quadraturei gauss

$$\int_a^b f(x) w(x) dx \approx \sum_{i=1}^n \Delta_i f(x_i) + R$$

Polinom	a	b	w
Chebšev	-1	1	$1/\sqrt{1-x^2}$
Légendre	-1	1	1
Lo Guerre	0	∞	e^{-x}
Hermite	$-\infty$	∞	$e^{-x^2/2}$

③ Fie formula de integr. $\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \sum_{i=1}^3 A_i f(x_i)$

unde x_1, x_2, x_3 nod pol. Cebâșev de grad 3

a) Să se det A_i a.f. gradul de valabilitate = 2

b) Calc. $\int_0^1 \frac{x^4 dx}{\sqrt{(1-x)x}}$ cu formula de noi sub.

a) $x_k = \cos \frac{2k+1}{6} \pi, k=0:2$

$$x_1 = \frac{\sqrt{3}}{2}$$

$$x_2 = 0$$

$$x_3 = -\frac{\sqrt{3}}{2}$$

$$f=1 \Rightarrow A_1 + A_2 + A_3 = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-1}^1 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$f=x \Rightarrow A_1 \cdot \frac{\sqrt{3}}{2} + A_3 \cdot \frac{-\sqrt{3}}{2} = \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx = 0 \quad (\text{fct. impară})$$

$$f=x^2 \Rightarrow A_1 \cdot \frac{3}{4} + A_3 \cdot \frac{3}{4} = \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx = 2 \int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx = \frac{\pi}{2}$$

$$A_1 = A_3$$

$$\frac{3}{2} A_1 = \frac{\pi}{2} \Rightarrow A_1 = \frac{\pi}{3} = A_3$$

$$A_2 = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

! Păr. grad n
de valabilitate $\Rightarrow$

$$A_i = \frac{\pi}{n+1}$$

$$b) \int_0^1 \frac{x^4}{\sqrt{x(1-x)}} dx = \int_0^1 \frac{x^{4-\frac{1}{2}}}{\sqrt{1-(\sqrt{x})^2}} dx = \int_0^1 \frac{x^{\frac{7}{2}}}{\sqrt{1-(\sqrt{x})^2}} dx =$$

$$= \int_0^1 \frac{(\sqrt{x})^7}{\sqrt{1-(\sqrt{x})^2}} dx$$

$$\int_0^1 \frac{x^4}{\sqrt{x(1-x)}} dx = \int_{-1}^1 \frac{\frac{(t+1)^4}{16}}{\sqrt{\frac{t+1}{2} \cdot \frac{1-t}{2}}} \cdot \frac{dt}{2} = \int_{-1}^1 \frac{1}{16} \cdot \frac{(t+1)^4}{\sqrt{1-t^2}} dt =$$

$$x = \frac{t+1}{2} \quad (\Rightarrow) \quad t = 2x-1$$

$$= \frac{1}{16} \int_{-1}^1 \frac{(t+1)^4}{\sqrt{1-t^2}} dt = \frac{1}{16} \cdot \frac{\pi}{3} \cdot \left(f\left(\frac{\sqrt{3}}{2}\right) + f(0) + f\left(-\frac{\sqrt{3}}{2}\right) \right) =$$

$$= \frac{\pi}{48} \cdot \left(\left(\frac{\sqrt{3}}{2}+1\right)^4 + 1 + \left(-\frac{\sqrt{3}}{2}+1\right)^4 \right) =$$

$$= \frac{\pi}{48} \cdot \frac{1}{16} \left[(\sqrt{3}+1)^4 + 16 + (1-\sqrt{3})^4 \right] = \frac{35\pi}{128}$$

④ Se consideră formula de integrare

$$\int_{-1}^1 f(x) dx = k(f(x_0) + f(x_1) + f(x_2))$$

Să se det k, x_0, x_1, x_2 a. i. formula să aibă gr. de volab. maximă

$$f=1 \quad \Rightarrow \quad k(1+1+1) = \int_{-1}^1 dx = 2 \quad \Rightarrow \quad \boxed{k = \frac{2}{3}}$$

$$f=x \quad \Rightarrow \quad k(x_0+x_1+x_2) = \int_{-1}^1 x dx = 0$$

$$f=x^2 \quad \Rightarrow \quad k(x_0^2+x_1^2+x_2^2) = \int_{-1}^1 x^2 dx = \frac{2}{3}$$

$$f=x^3 \quad \Rightarrow \quad k(x_0^3+x_1^3+x_2^3) = \int_{-1}^1 x^3 dx = 0$$

$$\begin{cases} x_0 + x_1 + x_2 = 0 \\ x_0^2 + x_1^2 + x_2^2 = 1 \\ x_0^3 + x_1^3 + x_2^3 = 0 \end{cases} \Rightarrow t^3 - \frac{1}{2}t = 0$$

$$t_0 = 0$$

$$t^2 - \frac{1}{2} = 0 \quad (\Leftrightarrow) \quad t_1 = \frac{\sqrt{2}}{2} \\ t_2 = -\frac{\sqrt{2}}{2}$$

$$\int_{-1}^1 f(x) dx = \frac{2}{3} \left(f(0) + f\left(\frac{\sqrt{2}}{2}\right) + f\left(-\frac{\sqrt{2}}{2}\right) \right)$$

⑤ Fie formula de integrare

$$\int_{-1}^1 f(x) dx = A_0 \cdot f(-1) + A_1 f(x) + A_2 f(x_2) + A_3 f(1)$$

grad de valab = 5

$$w(x) = (x+1)(x-1) = x^2 - 1$$

$$\pi(x) = (x-x_1)(x-x_2) = x^2 - 5x + 4$$

$$\int_{-1}^1 w(x) \cdot \pi(x) \cdot \pi^*(x) dx = 0$$

$$\pi^*(x) = ax + b$$

$$\int_{-1}^1 (x^2 - 1)(x^2 - 5x + 4)(ax + b) dx = 0 \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \int_{-1}^1 (x^2 - 1)(x^2 - 5x + 4) \cdot x dx = 0 \\ \int_{-1}^1 (x^2 - 1)(x^2 - 5x + 4) dx = 0 \end{cases}$$

$$\int_{-1}^1 (x^4 - 5x^3 + px^2 - x^2 + 5x - p)x \, dx = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \int_{-1}^1 x^5 - 5x^4 + (p-1)x^3 + 5x^2 - px \, dx = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow -\frac{25}{5} + \frac{25}{3} = 0 \quad \Rightarrow \quad p = 0$$

$$\int_{-1}^1 (x^4 - 5x^3 + px^2 - x^2 + 5x - p) \, dx = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \frac{2}{5} + \frac{2p}{3} - \frac{1}{3} - 2p = 0 \quad \Rightarrow \quad p = -\frac{1}{5}$$

$$t^2 - \frac{4}{5} = 0$$

$$t_{1,2} = \pm \sqrt{\frac{4}{5}}$$

$$f=1 \quad \Rightarrow \quad A_0 + A_1 + A_2 + A_3 = 2$$

$$f=x \quad \Rightarrow \quad -A_0 + A_1 \frac{\sqrt{5}}{5} + A_2 \frac{\sqrt{5}}{5} + A_3 = 0$$

$$f=x^2 \quad \Rightarrow \quad A_0 + A_1 \frac{1}{5} + A_2 \frac{1}{5} + A_3 = \frac{2}{3}$$

$$f=x^3 \quad \Rightarrow \quad -A_0 - A_1 \frac{\sqrt{5}}{25} + A_2 \frac{\sqrt{5}}{25} + A_3 = 0$$

$$\begin{cases} \frac{4A_1}{5} + \frac{4A_2}{5} = \frac{2}{3} & \Rightarrow \quad A_1 + A_2 = \frac{5}{3} \\ A_1 \sqrt{5} \left(\frac{1}{25} - \frac{1}{5} \right) + A_2 \sqrt{5} \left(\frac{1}{5} - \frac{1}{25} \right) = 0 & \Rightarrow \quad A_1 - A_2 = 0 \end{cases}$$

$$\Rightarrow A_1 = A_2 = \frac{5}{6}$$

$$A_0 = A_3 = 2 - \frac{5}{6} = \frac{1}{6}$$

$$\textcircled{6} \quad I = \int_{-\infty}^a e^x f(x) dx = - \int_{-\infty}^{-a} e^{-x} f(-x) dx =$$

$$x = -t$$

$$= \int_{-a}^{\infty} e^{-x} f(-x) dx = \int_0^{\infty} e^{-t-a} f(-t-a) dt =$$

$$x = t+a$$

$$= e^{-a} \int_0^{\infty} e^{-t} f(-t-a) dt = e^{-a} \sum_{i=1}^n \Delta_i f(-t_i - a)$$

t_i - r  d. pol. de Guerre
de grade n