

Metode iterative de rezolvare a
sistemelor de ecuatii

$$A \cdot x = b$$

$$(A \cdot x = b \Rightarrow x = A^{-1} \cdot b)$$

- dacă A nu e inversabilă sau prin inversare se produce instabilitate mare de rînd. $\Rightarrow$ folosim met. iterative

$x_0 = 0$ aprox a sol inițială

$$x_{n+1} = f(x_n, A, b)$$

$$\lim_{n \rightarrow \infty} F_n = F$$

$A =$

 $= D - E - F = N - P$

$$(N-P) \cdot \epsilon = b$$

$$N_x - P_x = b$$

$$N\hat{x}_{n+1} = P\hat{x}_n + b$$

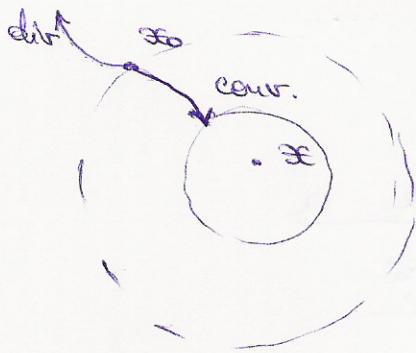
N-trebuie alesă converință pt. a fi univ. de inversat

$$x_{n+1} = N^{-1} \cdot P x_n + N^{-1} \cdot b$$

$$G = N \cdot P$$

Met. Joshi : $N=D$ $P=E+F$

Met. Gauss-Siedel: $N = D^{-1}F$ $P = E$



$$\|x_{n+1} - x_n\| < \varepsilon$$

$$\left| \begin{aligned} \|x_{n+1} - x_n\| &\leq \frac{\rho^n}{1-\rho} \|x_1 - x\| \\ \rho = \rho(G) &= \max_i \{|\lambda_i(G)|\} \end{aligned} \right.$$

$$Ax = b$$

① A -diagonal dominant

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \implies (C)$$

② $\rho(G) < 1$

ex:

1) $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ Jacobi

$$N = D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix} \rightarrow \text{diag}(\text{diag}(N)^{-1})$$

$$G = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -\frac{3}{2} \\ -\frac{1}{2} & -\lambda \end{vmatrix} = 0 \implies \lambda^2 - \frac{3}{4} = 0 \implies \lambda_{1,2} = \pm \frac{\sqrt{3}}{2} \implies$$

$$\Rightarrow |\lambda_1| = |\lambda_2| = \frac{\sqrt{3}}{2} < 1$$

→ Compararea celor 2 metode

$$\rho(G) < 1$$

$$\rho_J < \rho_{GS}$$

- cu cât val. spectrală e mai mică cu atât conv. mai rapid

2) Studiați conv. met. Jacobi și Gauss-Siedel ptr. urm. matrice (ptr. ce val. ale lui a conv.)

$$A = \begin{bmatrix} 1 & 0 & a \\ a & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

a) Jacobi

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & -a \\ -a & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -a \\ -a & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a \\ -\frac{a}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

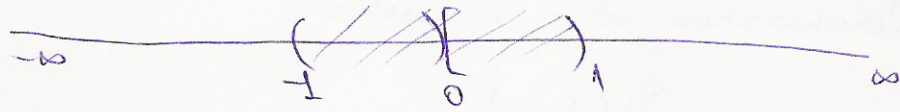
$$\begin{vmatrix} -\lambda & 0 & -a \\ -\frac{a}{2} & -\lambda & \frac{1}{2} \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 - \lambda a = 0 \Rightarrow \lambda(\lambda^2 + a) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm \sqrt{-a} \quad \Bigg| \Rightarrow a \in (-1, 1)$$

$$\rho(G) < 1$$

$$\text{Dacă } a \geq 0 \Rightarrow |\lambda_{2,3}| = \sqrt{a} \Rightarrow a \in [0, 1)$$

$$\text{Dacă } a < 0 \Rightarrow \sqrt{-a} < 1 \Rightarrow a \in (-1, 0)$$



b) Gauss-Siedel

$$N = \begin{bmatrix} 1 & 0 & 0 \\ a & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\det N = 2$$

$$P = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^* = \begin{bmatrix} 2 & -a & 0 \\ -a & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \Rightarrow N^{-1} = \begin{bmatrix} 1 & -a/2 & 0 \\ -a/2 & 1/2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & -a/2 & 0 \\ -a/2 & 1/2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & \frac{a^2}{2} + \frac{1}{2} \\ 0 & 0 & -a \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 0 & -a \\ 0 & -\lambda & \frac{a^2}{2} + \frac{1}{2} \\ 0 & 0 & -a - \lambda \end{vmatrix} = 0 \Rightarrow -\lambda^2(a + \lambda) = 0 \Rightarrow \begin{cases} \lambda_{1,2} = 0 \\ \lambda_3 = -a \end{cases}$$

$$|a| < 1 \Rightarrow a \in (-1, 1)$$

$$\rho_f(G) = \sqrt{|a|}$$

$$\rho_{GS}(G) = |a|$$

$$1) \quad A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$$

$$x_{n+1} = N^{-1} \cdot P \cdot x_n + N^{-1} \cdot b$$

$$x_1 = \begin{bmatrix} 0 & -\frac{3}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0 & -\frac{3}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

$$x_3 = \dots$$

Interpolation

x	x_1	x_2	...	x_n
f	f_1	f_2	...	f_n

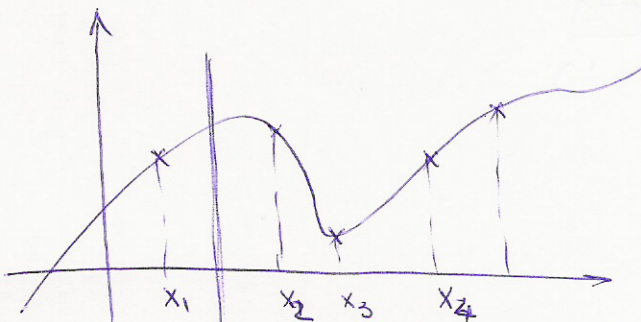
$$f: [x_1, x_n] \rightarrow \mathbb{R}$$

$$f(x_i) = f_i$$



$$\mathbb{R}_{n-1}[X]$$

$$P(x_i) = f_i$$



I Lagrange

$$P(x) = \sum_{i=1}^n f_i \cdot \prod_{\substack{j=1 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

ex:

1)

x	a	0	1	b
y	y_a	y_0	y_1	y_b

$$P(x) = y_a \cdot \frac{(x-0)(x-1)(x-b)}{(a-0)(a-1)(a-b)} + y_0 \cdot \frac{(x-a)(x-1)(x-b)}{(0-a)(0-1)(0-b)} +$$

$$+ y_1 \cdot \frac{(x-a)(x-0)(x-b)}{(1-a)(1-0)(1-b)} + y_b \cdot \frac{(x-a)(x-0)(x-1)}{(b-a)(b-0)(b-1)}$$

2)

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

$$P(x) = 1 \cdot \frac{(x+\frac{1}{2})(x-0)(x-\frac{1}{2})(x-1)}{(-1+\frac{1}{2})(-1-0)(-1-\frac{1}{2})(-1-1)} + \frac{1}{2} \cdot \frac{(x+1)(x-0)(x-\frac{1}{2})(x-1)}{(\frac{1}{2}+1)(\frac{1}{2}-0)(\frac{1}{2}-\frac{1}{2})(\frac{1}{2}-1)} +$$

$$+ 0 \cdot \frac{(x+1)(x+\frac{1}{2})(x-\frac{1}{2})(x-1)}{(0+1)(0+\frac{1}{2})(0-\frac{1}{2})(0-1)} + \frac{1}{2} \cdot \frac{(x+1)(x+\frac{1}{2})(x-0)(x-1)}{(\frac{1}{2}+1)(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-0)(\frac{1}{2}-1)} +$$

$$+ 1 \cdot \frac{(x+1)(x+\frac{1}{2})(x-0)(x-\frac{1}{2})}{(1+1)(1+\frac{1}{2})(1-0)(1-\frac{1}{2})}$$

II Newton

$$P(x) = F_0[x] + F_1[x_1, x_2](x-x_1) + F_2[x_1, x_2, x_3](x-x_1)(x-x_2) + \dots + F_{n-1}[x_1, x_2, \dots, x_{n-1}](x-x_1)(x-x_2)\dots(x-x_{n-1})$$

$$F_0[x] = f_i$$

$$F_1[x_i, x_j] = \frac{F_0[x_j] - F_0[x_i]}{x_j - x_i}$$

$$F_p[x_1, x_2, \dots, x_{p+1}] = \frac{F_{p-1}[x_1, x_2, \dots, x_p] - F_{p-1}[x_2, x_3, \dots, x_{p+1}]}{x_1 - x_{p+1}}$$

ex:

1)

a	y_a	}	$\frac{y_0 - y_a}{0 - a}$	}	$\frac{\frac{y_1 - y_0}{1 - 0} - \frac{y_0 - y_a}{0 - a}}{1 - a}$	
0	y_0		$\frac{y_1 - y_0}{1 - 0}$			
1	y_1		$\frac{y_b - y_1}{b - 1}$			
b	y_b		$\frac{y_b - y_1}{b - 1} - \frac{y_1 - y_0}{1 - 0}$			
						$\frac{\frac{y_b - y_1}{b - 1} - \frac{y_1 - y_0}{1 - 0}}{b - a}$

F_1

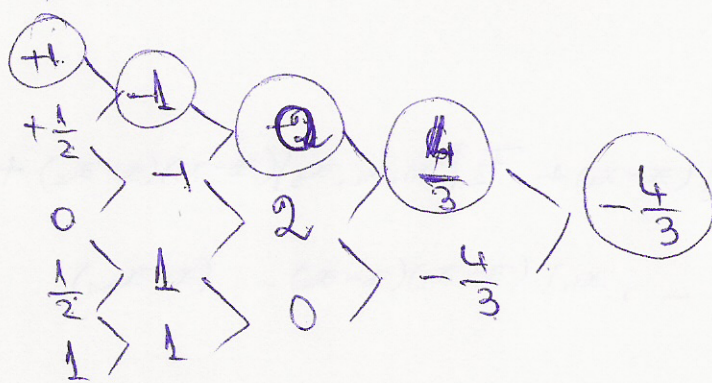
F_2

$\frac{\frac{y_1 - y_0}{1 - 0} - \frac{y_0 - y_a}{0 - a}}{1 - a}$	$-$	$\frac{\frac{y_b - y_1}{b - 1} - \frac{y_1 - y_0}{1 - 0}}{b - a}$
$\frac{\frac{\frac{y_1 - y_0}{1 - 0} - \frac{y_0 - y_a}{0 - a}}{1 - a} - \frac{\frac{y_b - y_1}{b - 1} - \frac{y_1 - y_0}{1 - 0}}{b - a}}{a - b}$		

F_3

C_4

$$P(x) = C_1 + C_2(x-a) + C_3(x-a)x + C_4(x-a)x(x-1)$$

$$\begin{array}{c} -1 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{array}$$


Erreos

$$E(x) = \left| \prod_{i=1}^n (x - x_i) \cdot f_n[x_1, x_2, \dots, x_n, x] \right| \leq$$

$$\leq \left| \prod_{i=1}^n (x_i - x_j) \cdot \frac{f^{(n)}(\xi)}{(n-1)!} \right| \quad \xi \in [x_1, x_n]$$

$$E(x) \leq \left| \left(x^2 - \frac{1}{4}\right) (x^2 - 1) \cdot x \cdot \frac{f^{(5)}(\xi)}{24} \right|, \quad \xi \in [-1, 1]$$

Coef. pd Lagrange

$$P(x) = \sum_{i=1}^n f_i \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(x - x_j)}{(x_j - x_i)} = \sum_{i=1}^n c_i \prod_{\substack{j=1 \\ j \neq i}}^n (x - x_j)$$

$$C_i = \frac{f_i}{\frac{n}{\sum_{j=1}^c (x_i - x_j)}}$$

In MATLAB:

```
function c = coefLagrange (f, x)
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```
    n = length(x);
```

```
    c(1:n) = f(1:n);
```

```
    for i = 1:n
```

```
        p = 1
```

```
        for j = 1:n
```

```
            if j ~ i
```

```
                p = p * (x(i) - x(j));
```

```
            end
```

```
        end
```

```
        c(i) = c(i) / p;
```

```
    end
```