Grover's algorithm


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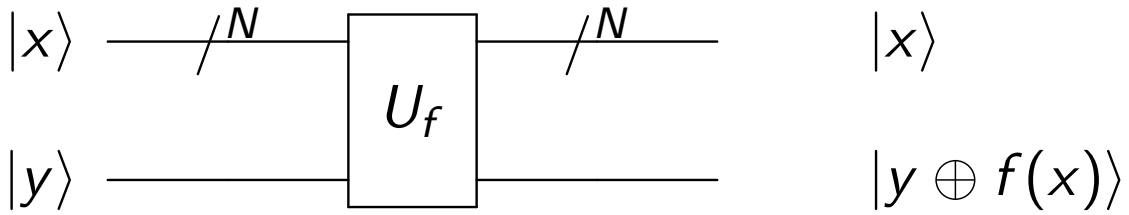
It's proven optimal!
(in the quantum case)

Unstructured search



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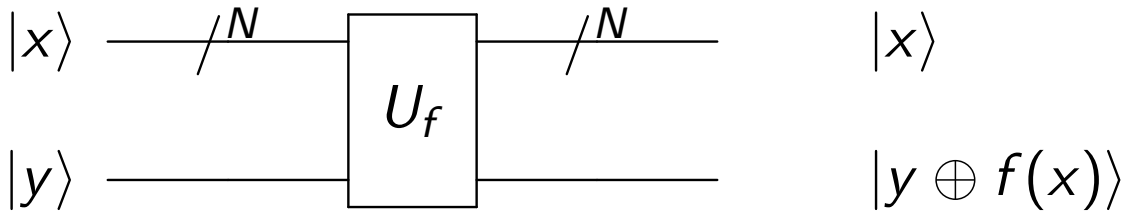

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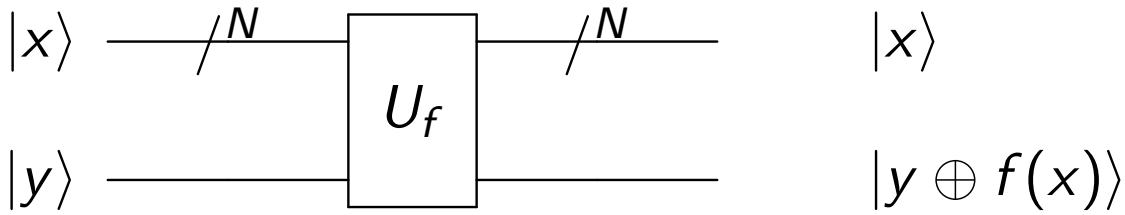

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First input $\sum_{x \in \{0,1\}^N} \frac{1}{\sqrt{2^N}} |x\rangle$

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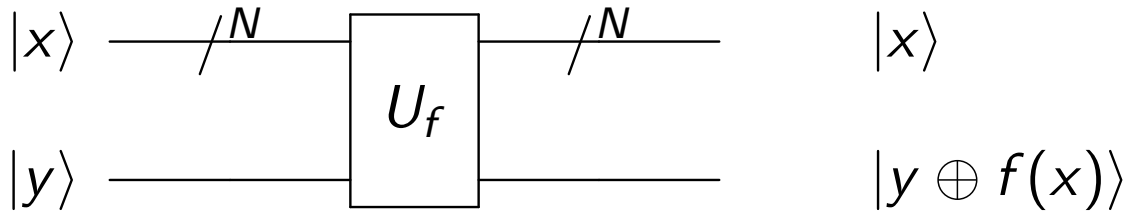

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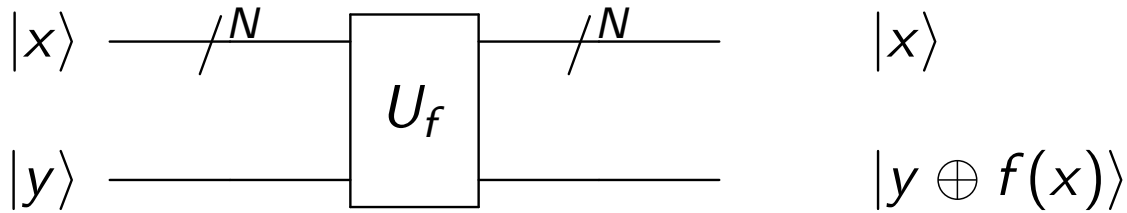

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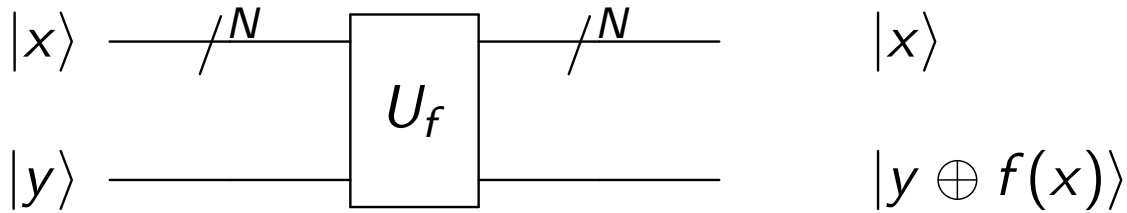

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So the first state is actually:

$$\frac{1}{\sqrt{2^N}}(\sum_{x\neq s} |x\rangle - |s\rangle)$$

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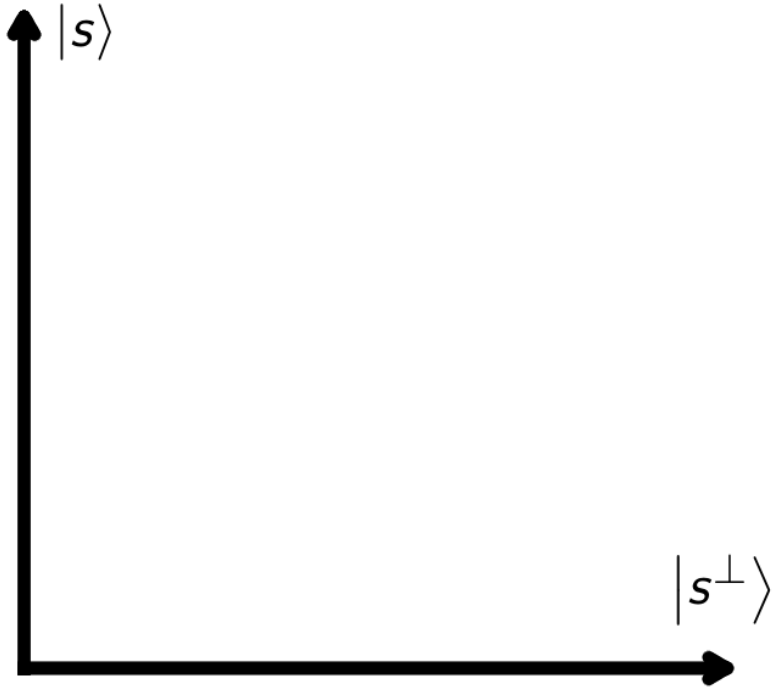

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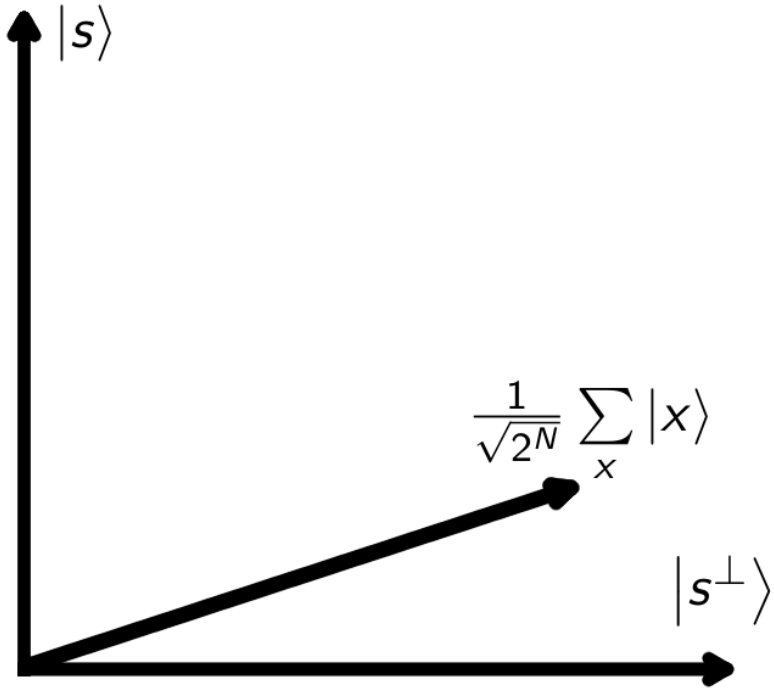
In between them is $\frac{1}{\sqrt{2^N}} \sum_{x} |x\rangle$

$$\frac{\sqrt{2^N-1}}{\sqrt{2^N}} \left|s^{\perp}\right\rangle + \frac{1}{\sqrt{2^N}} |s\rangle$$



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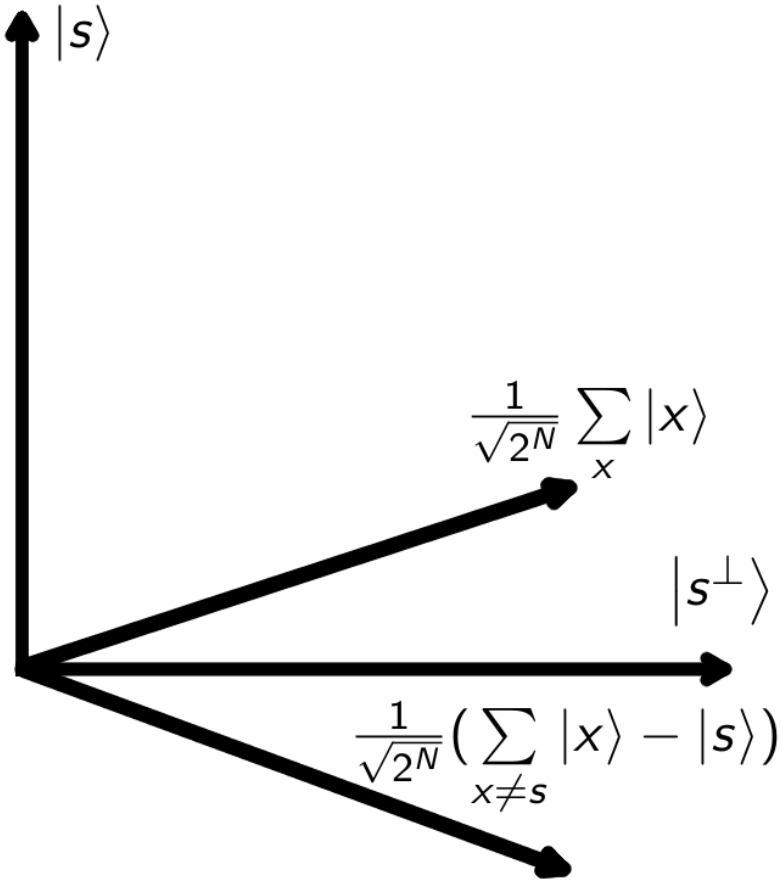

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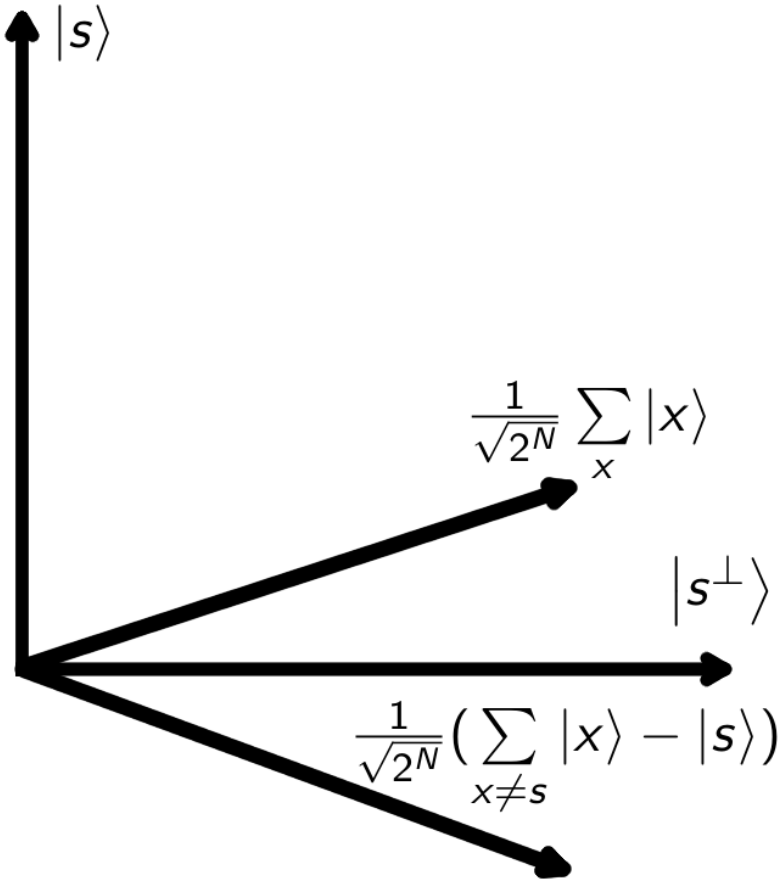

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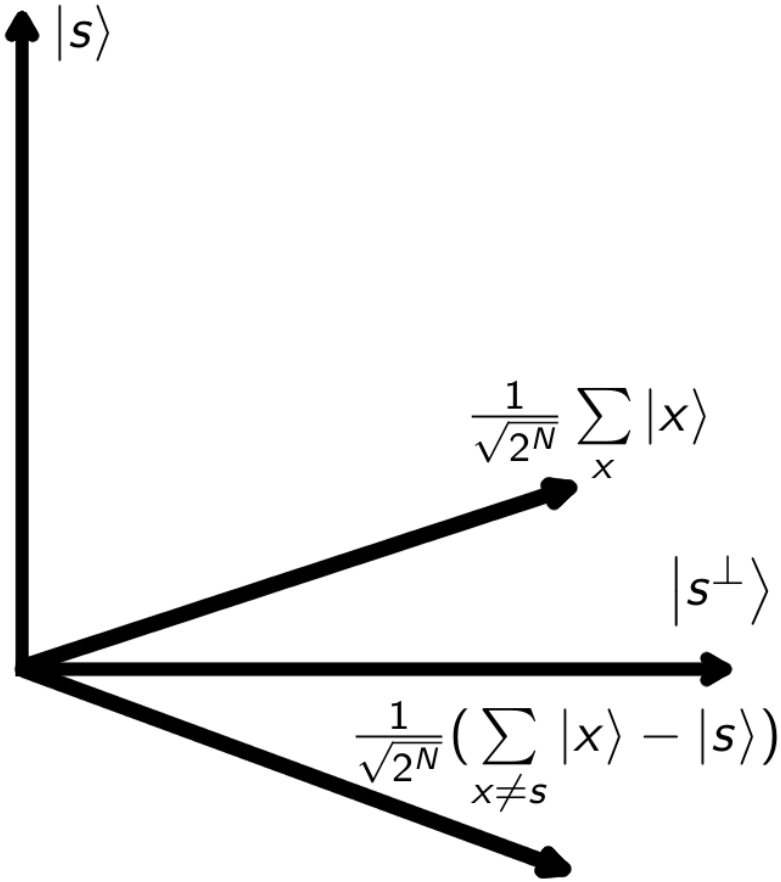
And ended with $a\left|s^\perp\right\rangle - b|s\rangle$



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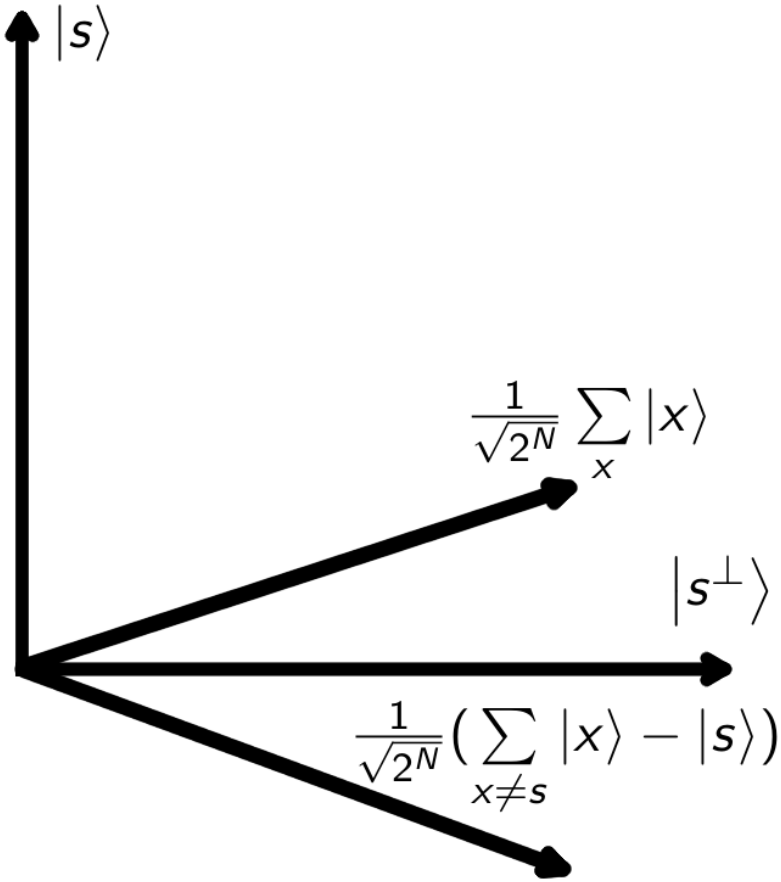

Reflection!

Unstructured search

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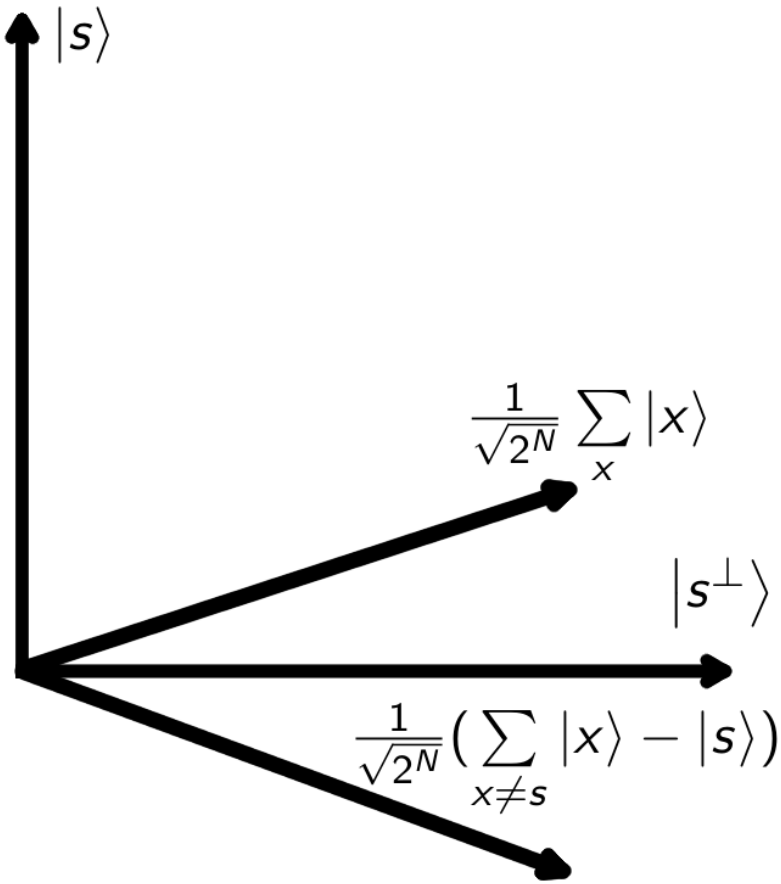
We've reflected our input with respect to $\left|s^{\perp}\right\rangle$



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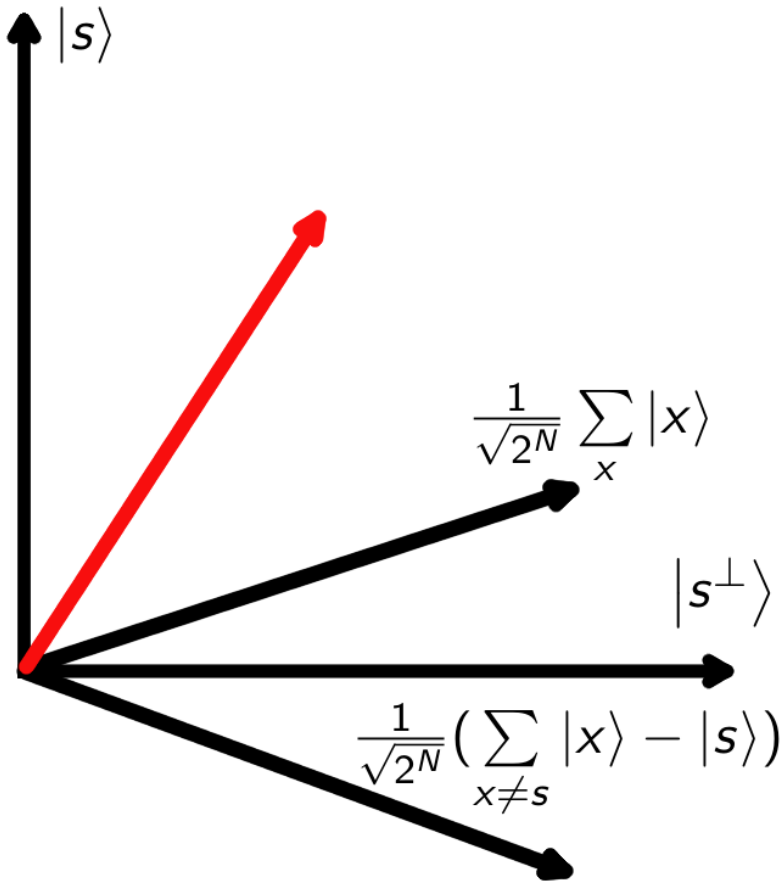

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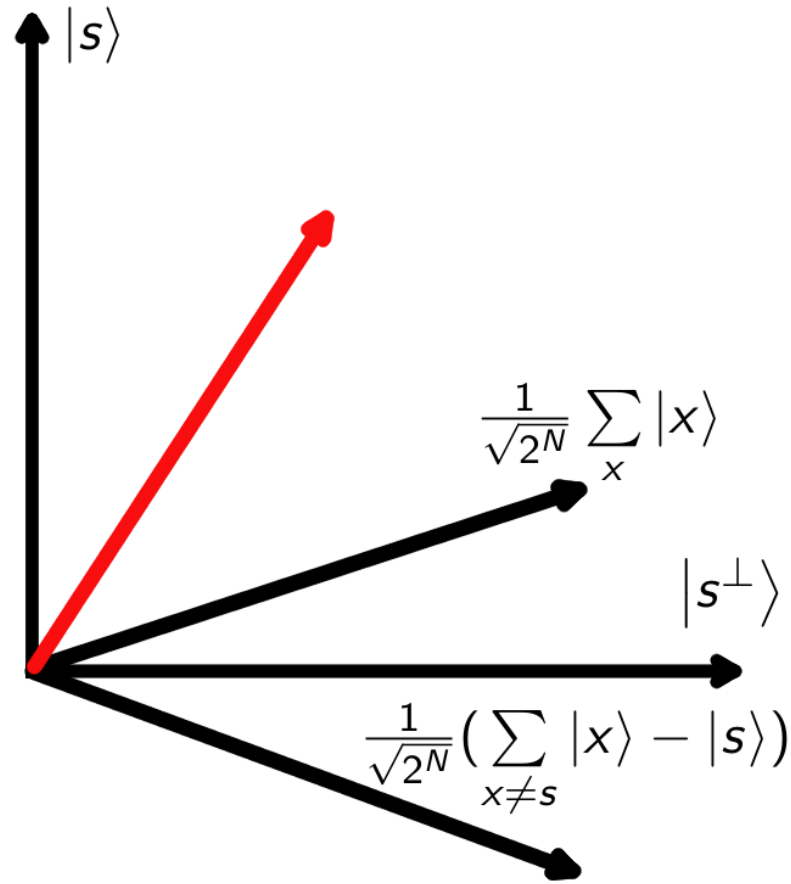
We've reflected our input with respect to $|s^{\perp}\rangle$

Reflect back, with respect to $\frac{1}{\sqrt{2^N}} \sum_x |x\rangle$

This brings us closer to $|s\rangle$!



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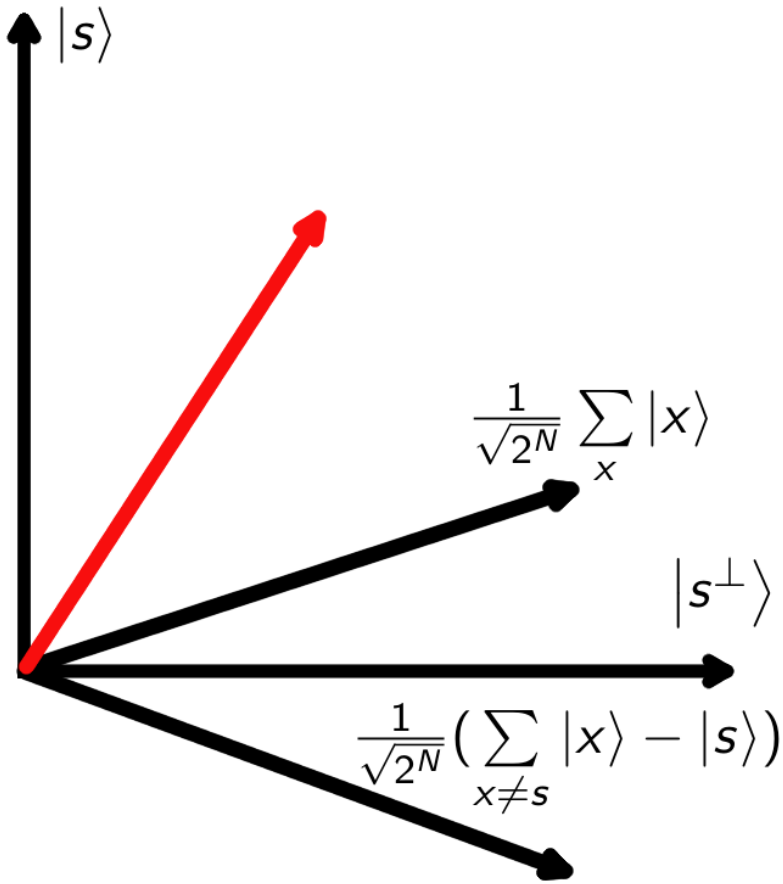
Repeat until we reach $|s\rangle$



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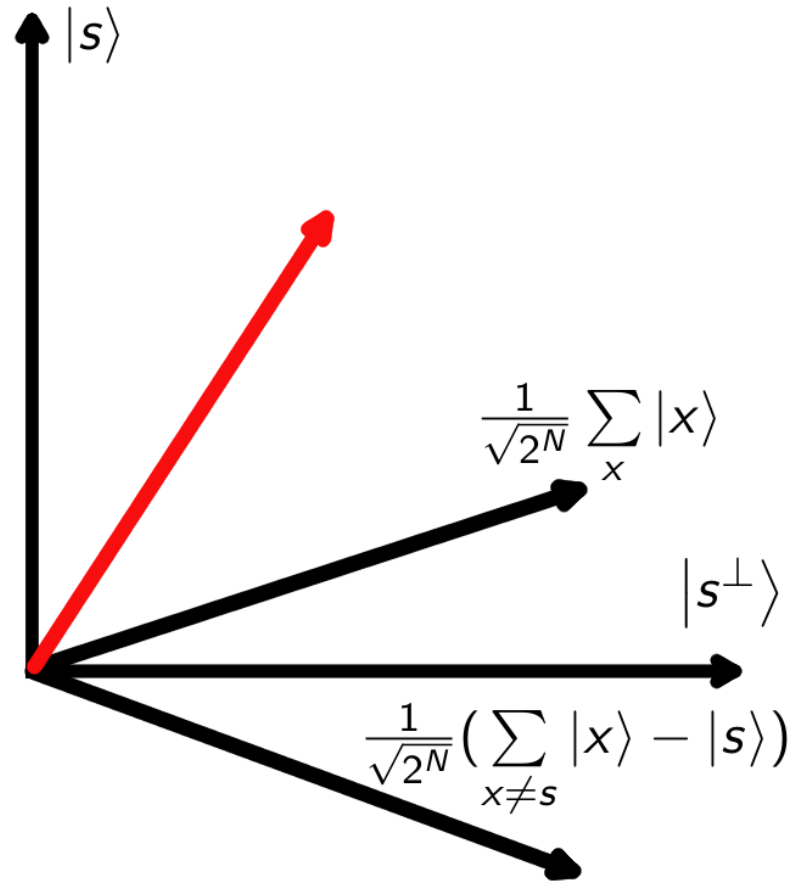

How many times?

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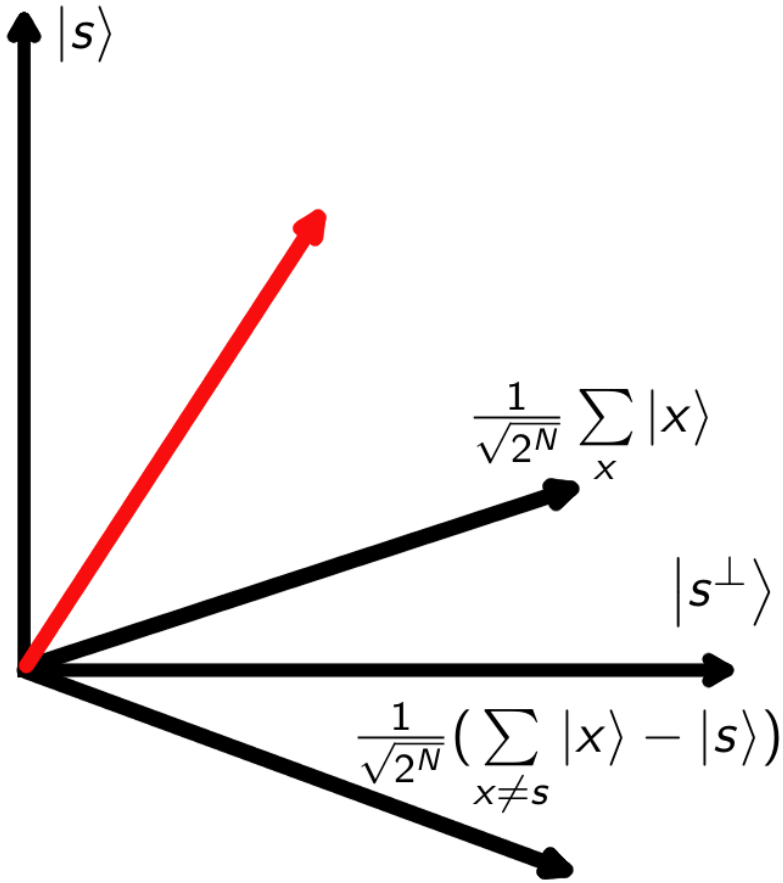
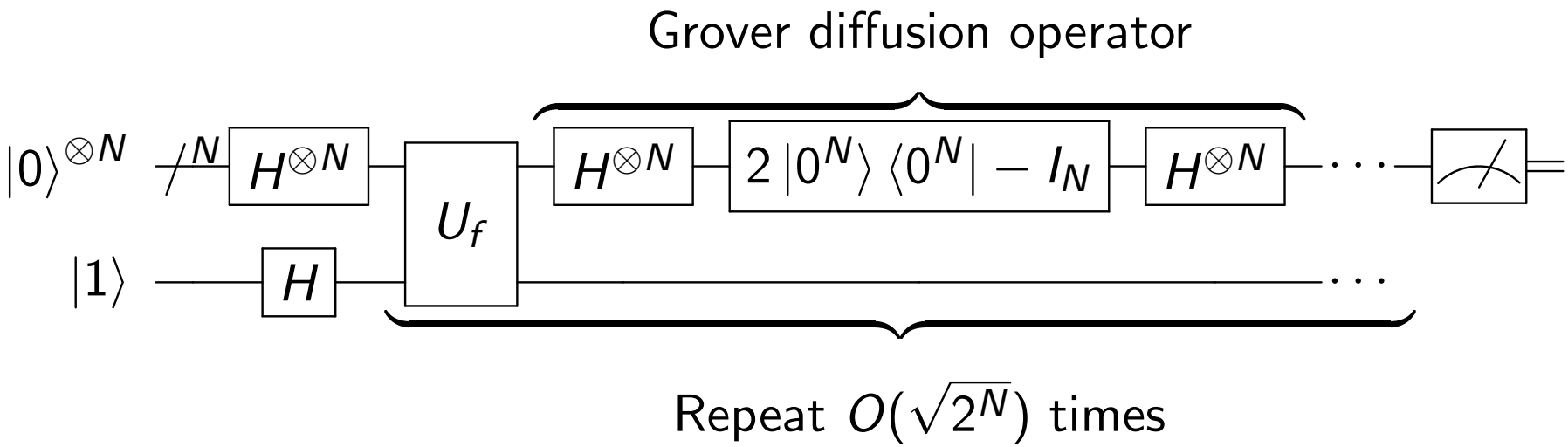


How many times?

$$O(\sqrt{2^N}) = O(\sqrt{n})$$

Grover's algorithm



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- NP-hard problems (travelling salesman, Hamiltonian cycle)
- Variants of Grover are useful for other problems (e.g. finding collisions)

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Assume $N = 66$

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Quantum computer takes 100 days to solve problem!