Quantum Computation and Cryptography
Day 4

Alexandru Gheorghiu

The University of Edinburgh

Part II
Quantum algorithms

Second problem - Simon's problem

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How many calls to f do we need, to determine type with high probability?

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What is the probability that *any* two of them is a collision pair?

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This is actually optimal, so: #calls to f $= \Omega(2^{N/2})$

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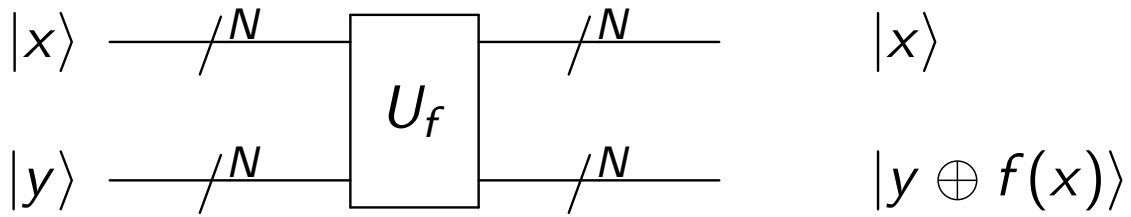

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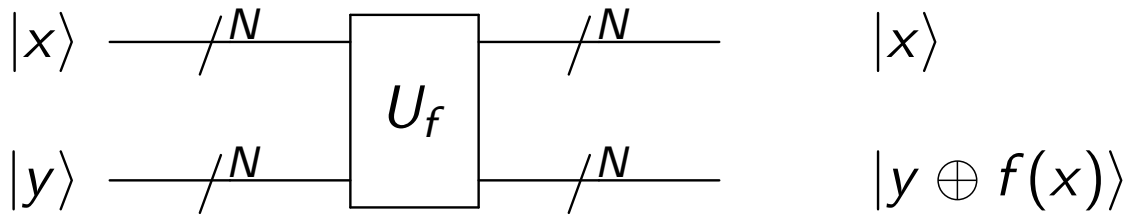

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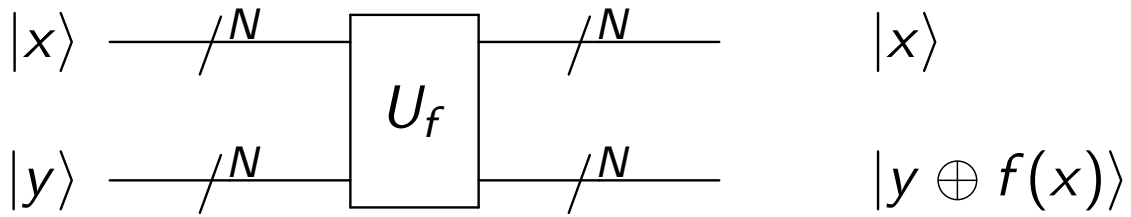

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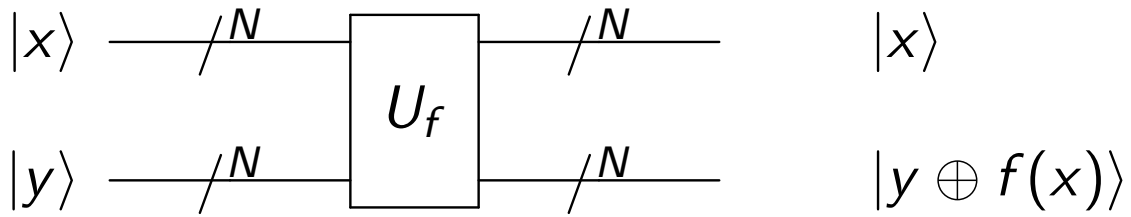

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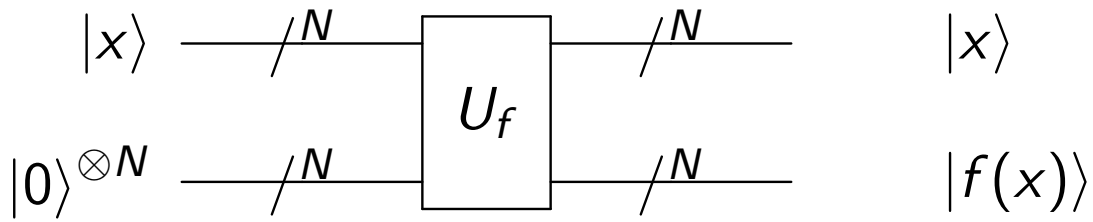

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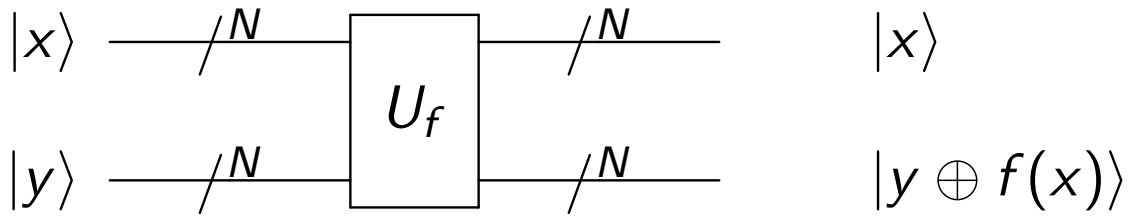

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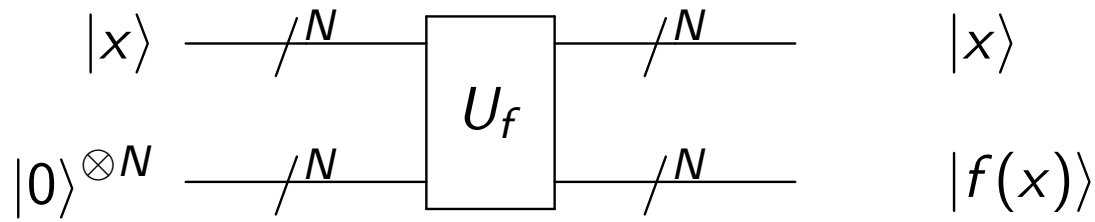

Call the upper part register 1 and the lower part register 2

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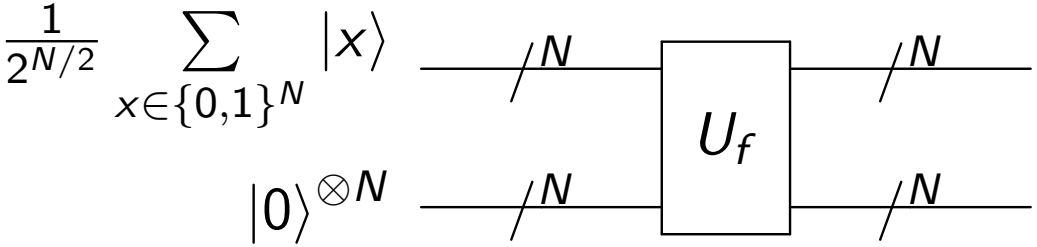

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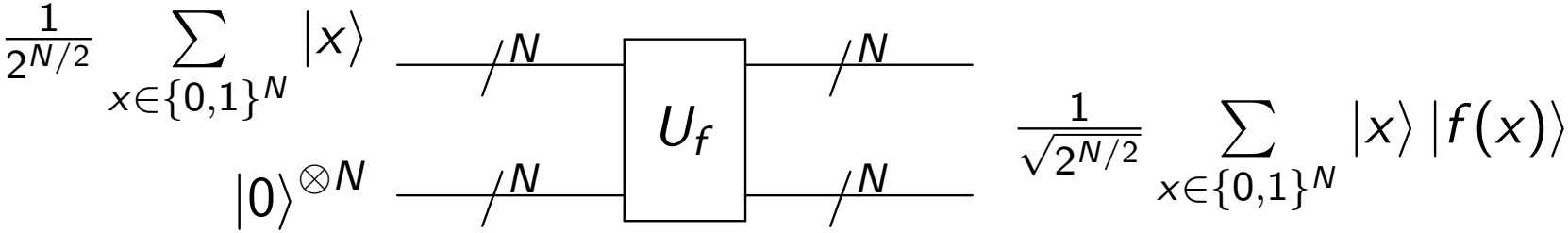

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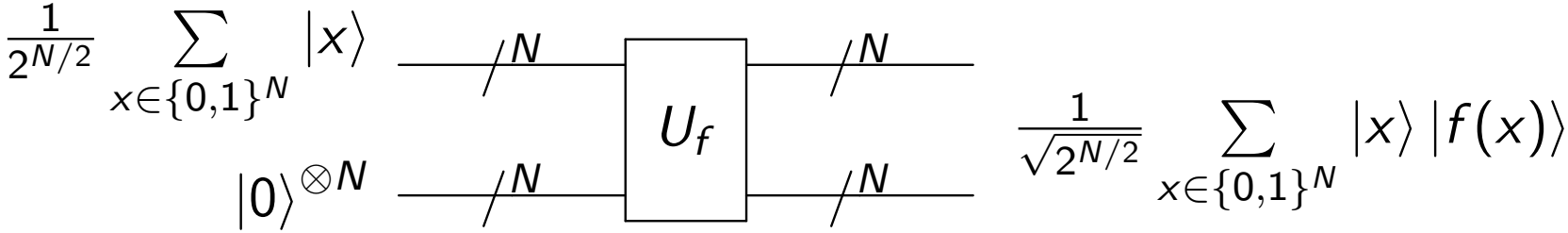

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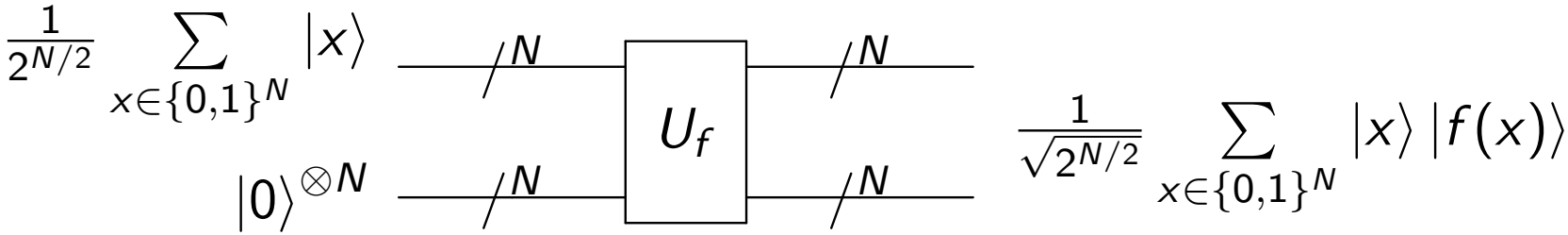

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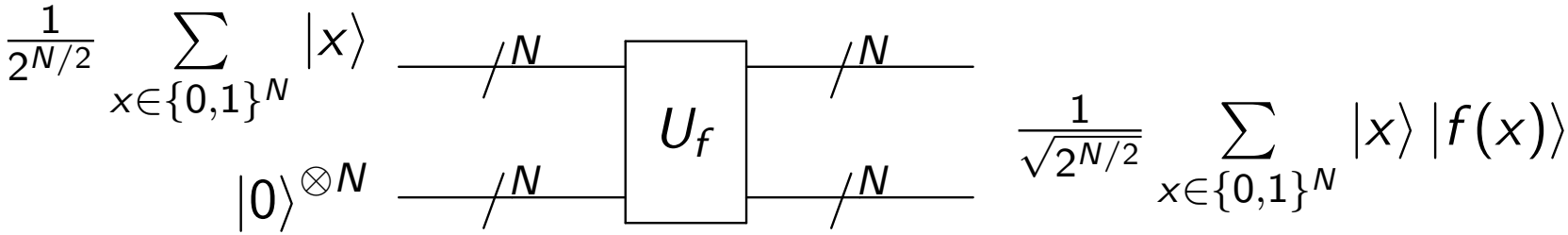

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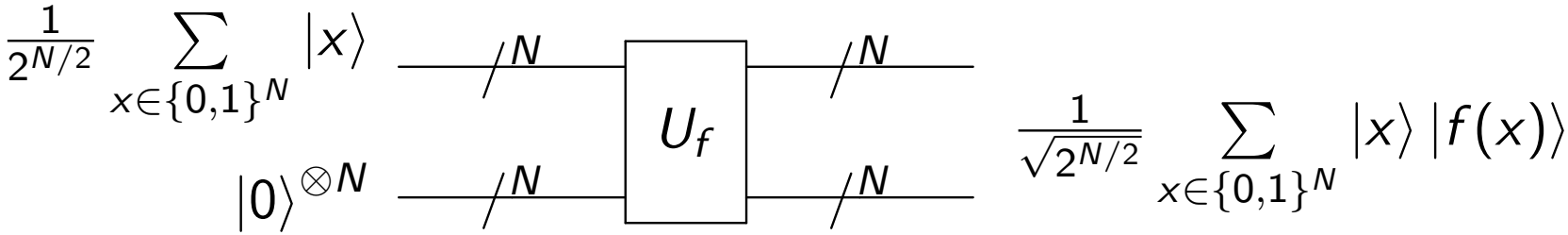

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We can solve this system (efficiently) and recover s

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We can solve this system (efficiently) and recover s

Only required $O(N)$ queries to f

Quantum case

Repeating $O(N)$ times, with high probability, we get a system of equations:

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Exponential speed-up!

Questions so far?

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Time to break some classical crypto...

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But first a short primer on RSA

Public-key crypto 101

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RSA, ElGamal, ECC are *public-key* cryptosystems

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What's that?

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Say you want to buy an app

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What's that?

Say you want to buy an app

You need to securely send your payment details (card number etc)

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Asymmetric encryption idea:

- There are 2 keys: K_1, K_2
- Messages encrypted with K_1 can only be decrypted with K_2

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4. Sends encrypted content to Appstore

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What an eavesdropper can know: PK, encrypted messages

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What an eavesdropper can know: PK, encrypted messages

Eavesdropper should not be able to decrypt (even partially) encrypted messages

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A bit more precise...

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Do such functions exist?

We don't know! But it's strongly believed that they do

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Suppose we pick two odd prime numbers P and Q

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$\{1, 2, 4, 7, 8, 11, 13, 14\}$

For some group element g, we can efficiently find it's inverse, i.e. h such that $hg = 1 \ mod \ N$

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The attacker knows N, e, C, but would need d to recover M

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Well just factor N and recover P and Q

Factoring

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But there is a quantum algorithm! (Shor's algorithm)

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What is (likely to be) their least common multiplier?

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Input: g, N
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Shor's algorithm

Simon: find s for which $f(x) = f(x \oplus s)$

We're trying to do same thing, but over a different group!

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(we're using $T = log_2 N$ qubits for $|x\rangle$, same for $|f(x)\rangle$)

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Where ω is the 2^T'th root of unity

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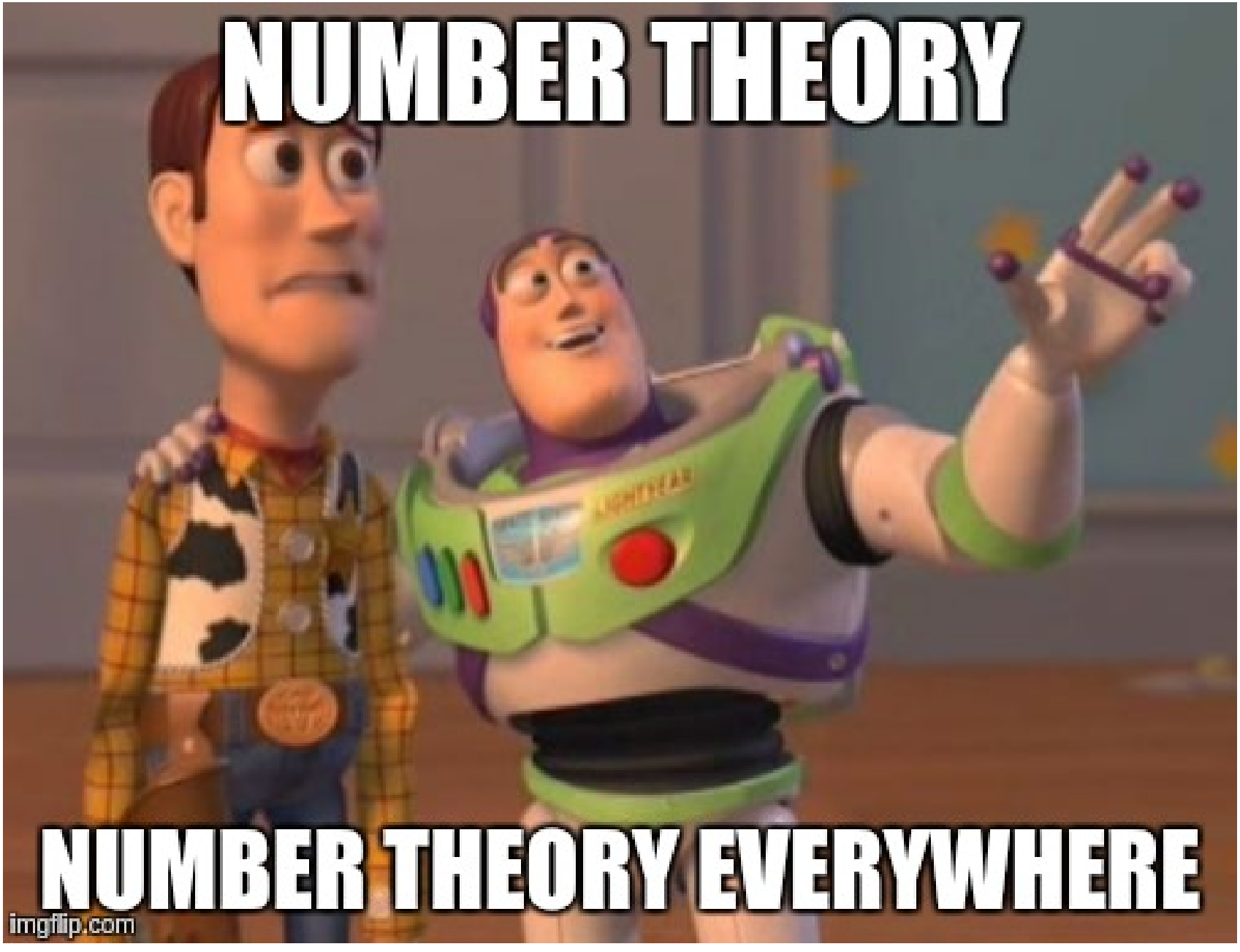
We can find k!


NUMBER THEORY
NUMBER THEORY EVERYWHERE
imgflip.com

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So they are insecure against a quantum computer!

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Problems believed to be hard even for quantum computers

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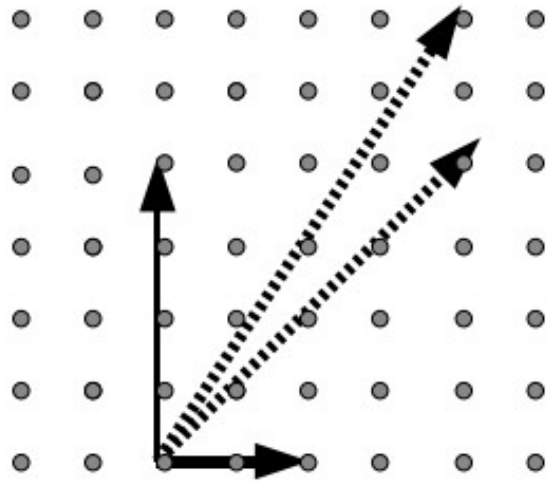
Good candidates for crypto: lattice problems



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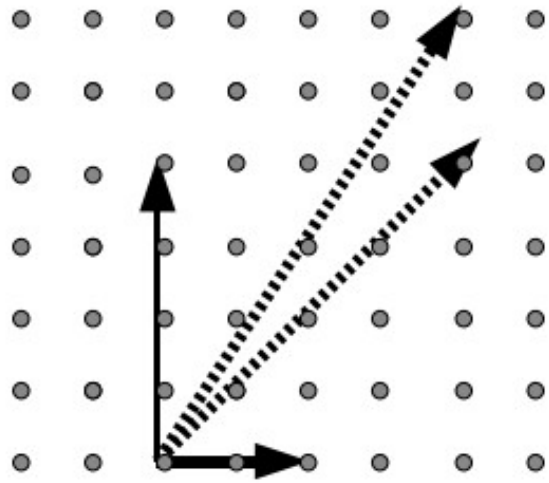

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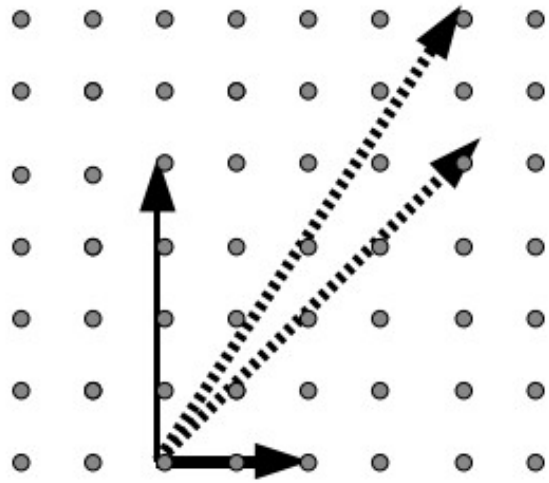

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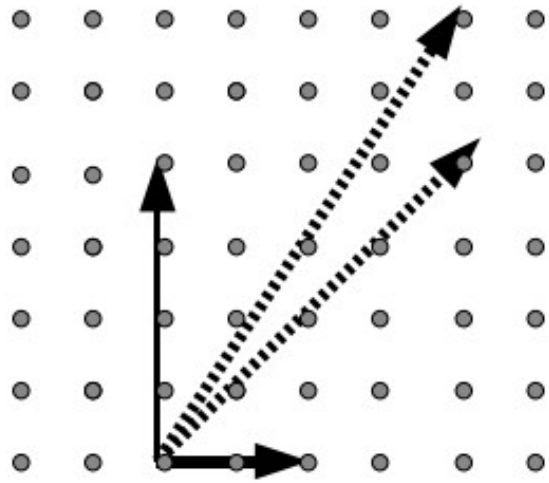

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Shortest vector problem:

Input: a basis to a lattice

Output: shortest vector in the lattice

Food for thought

A quantum computer can do anything a classical computer can do
(at least equally as efficient)

To simulate a classical computer, it can work only in the
computational basis
(without superpositions)

But all operations would still have to be reversible
(because of unitarity)

So can't we just make an efficient circuit for multiplying two
numbers and run it in reverse?

Useful resources and references

- **Pollard's rho algorithm** - http://en.wikipedia.org/wiki/Pollard%27s_rho_algorithm
- **A nice overview of quantum algorithms** - http://www.nature.com/articles/npjqi201523
- **Efficient circuit implementation of Shor's algorithm** - http://arxiv.org/pdf/quant-ph/0205095v3.pdf
- **Scott Aaronson explaining Simon, Shor, RSA and other stuff :)** - http://web.de.mit.edu/public/courses/6/6.045/2015spring/
- **Breaking symmetric key crypto using Simon's algorithm** - http://arxiv.org/abs/1602.05973
- Image on slide 11 (buying an app) - http://cdn2.iconfinder.com/data/icons/shopping-e-commerce-3/512/mobile-finger-512.png
- Image on slide 23 (lattice) - http://courses.cs.washington.edu/courses/cse522/04au/notes/lect18.pdf