Quantum Computation and Cryptography
Day 4

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Part I
Quantum algorithms

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Greek letters for any states (computational basis or not)
E.g. $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

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E.g.: A succeeds with probability $3/4$ and its running time is $O(N^2)$ and $\Omega(N)$

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This includes all classical polynomial time computation

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- Period finding, order finding, searching, factoring, discrete logarithm, optimization, "solving" linear systems etc

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We don't want p to be arbitrarily close to 0

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p could be arbitrarily (e.g. exponentially) close to $1/2$

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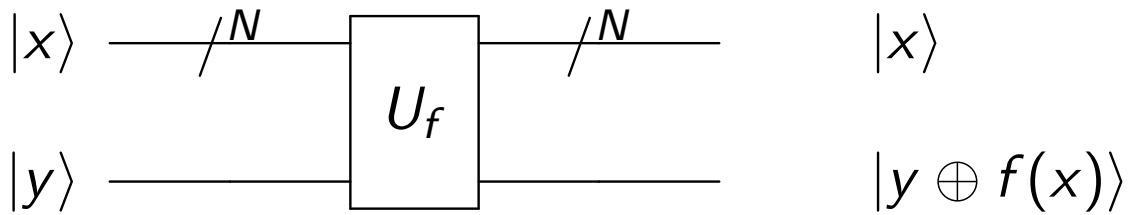

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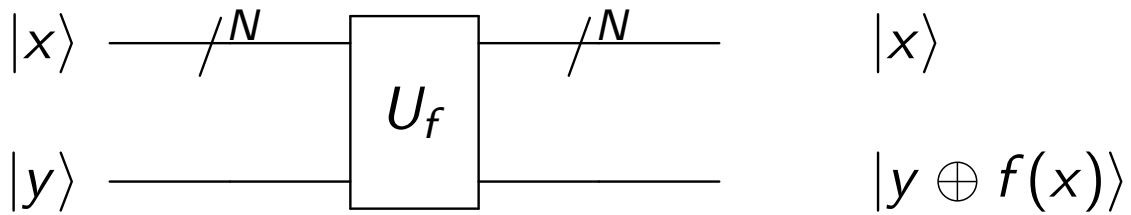

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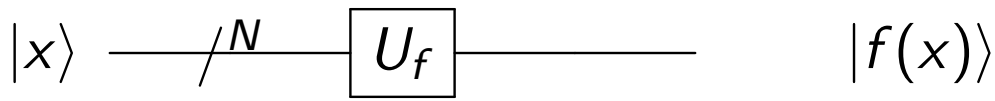

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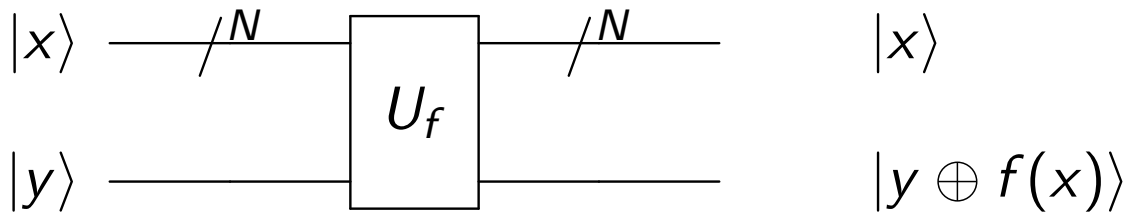

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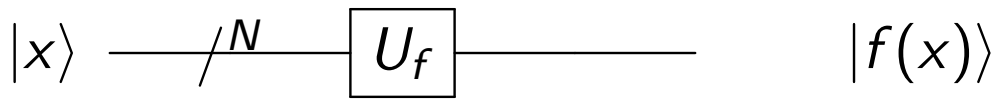

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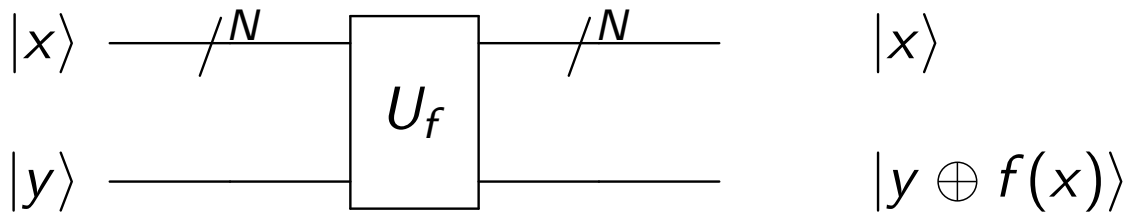
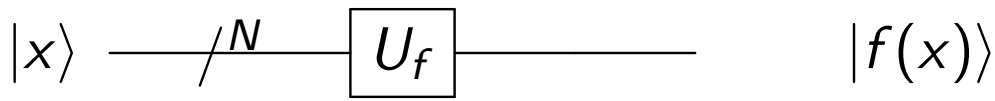


Why not...



Not unitary!

Number of inputs must equal number of outputs!

Quantum case

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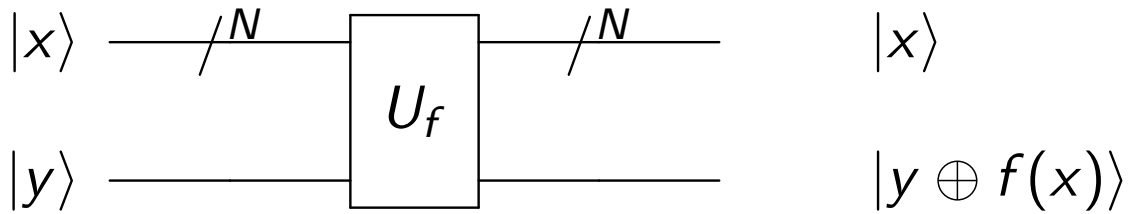

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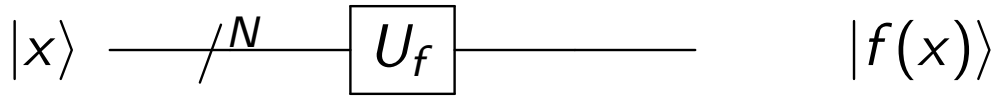

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Must be **reversible**!

An example

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You've already seen an example of balanced U_f!

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Suppose $f : \{0, 1\} \to \{0, 1\}$, $f(x) = x$

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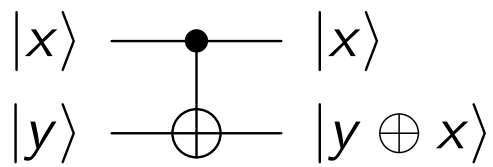

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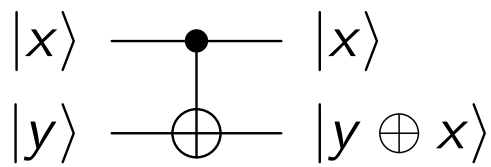

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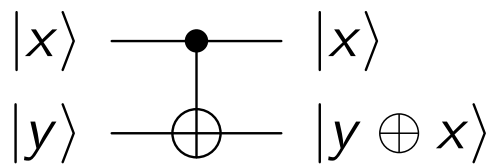

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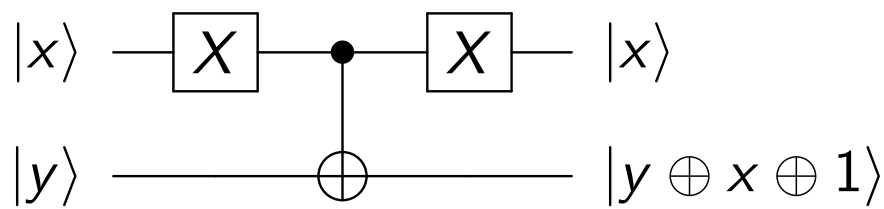

Simplified version

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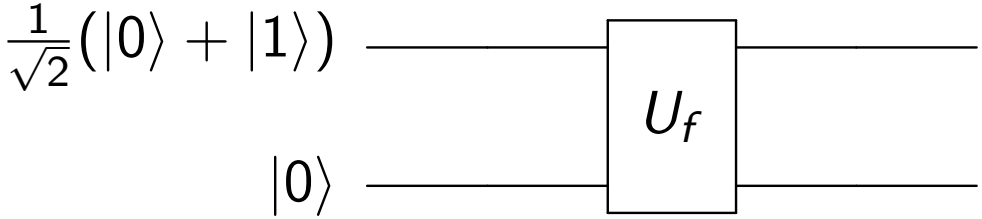

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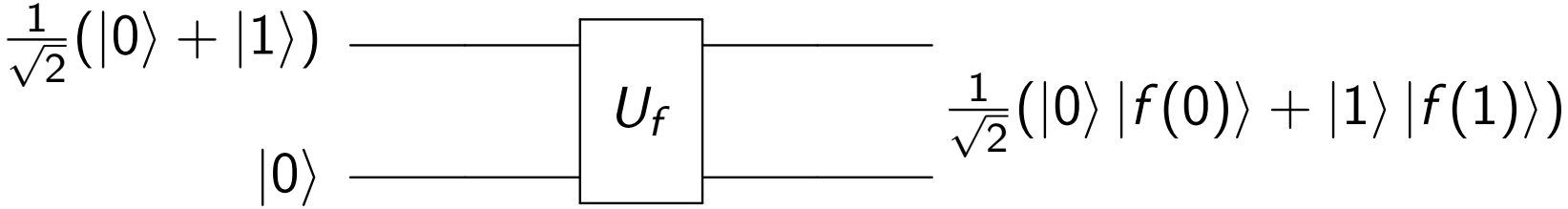

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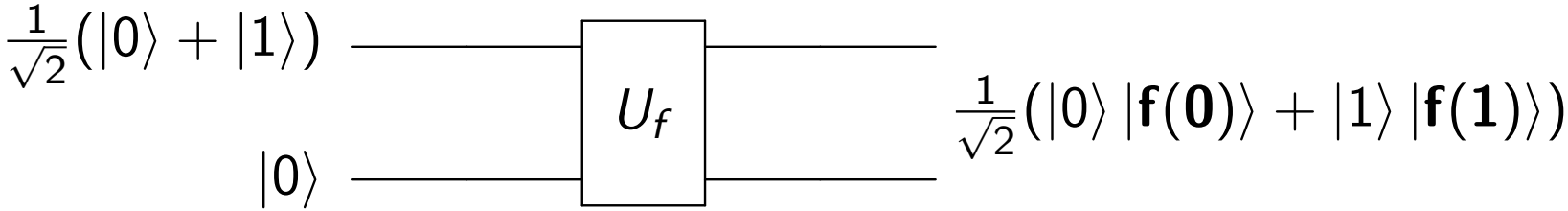

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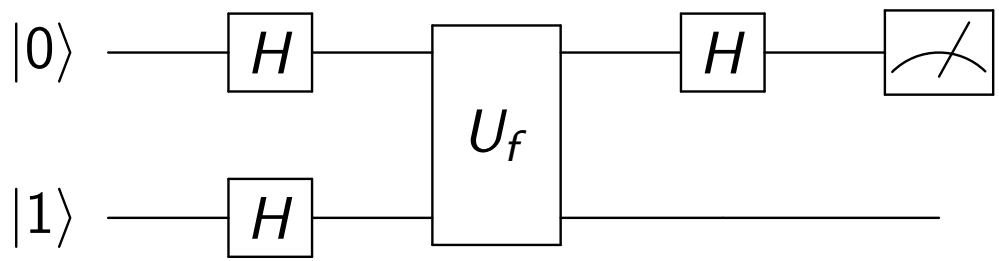

Simplified version

$$\text{Constant: } \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$\text{Balanced: } \pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

If we apply H to first qubit we get...

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$$\text{Balanced: } \pm|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

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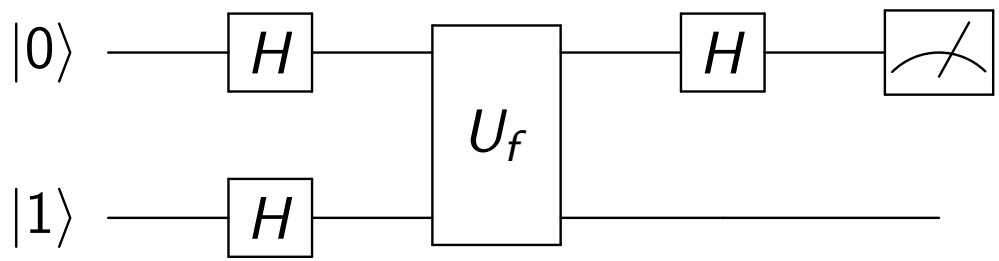


Deutsch's algorithm

Before moving on... questions and discussions

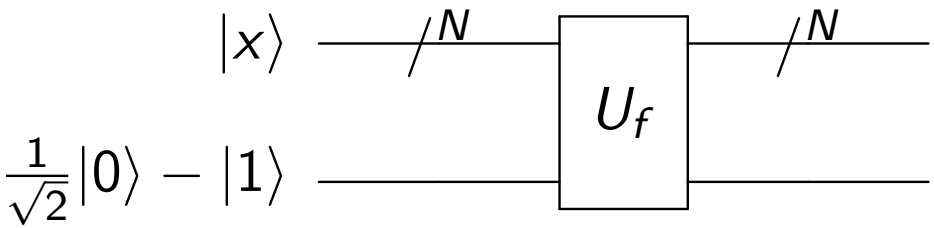
General version

General version

Back to the general case

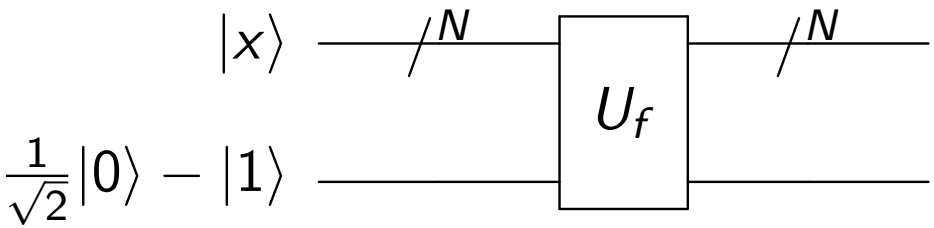
General version

Back to the general case



General version

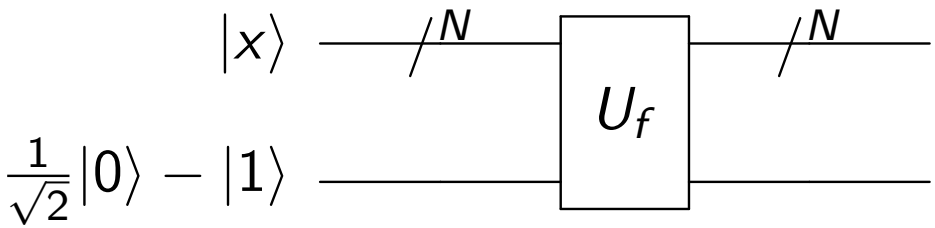
Back to the general case



What is the output when $|x\rangle$ is an arbitrary basis state?

General version

Back to the general case



$$\frac{1}{\sqrt{2}}(|x\rangle\,|f(x)\rangle - |x\rangle\,|1 \oplus f(x)\rangle)$$

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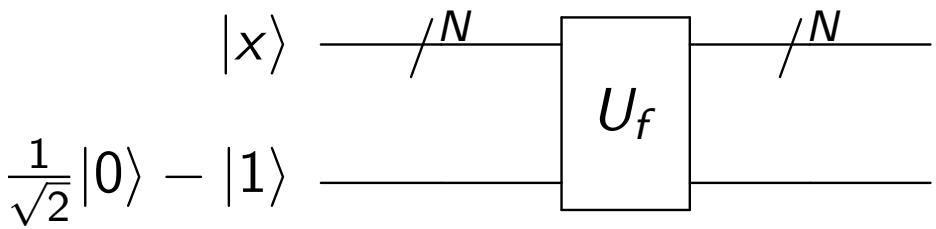

$$|x\rangle \otimes \frac{1}{\sqrt{2}}(|f(x)\rangle - |1 \oplus f(x)\rangle)$$

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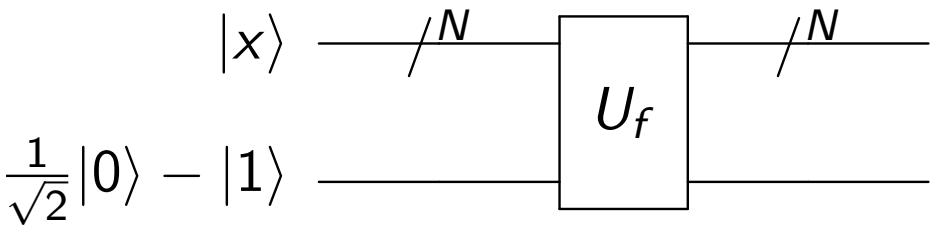

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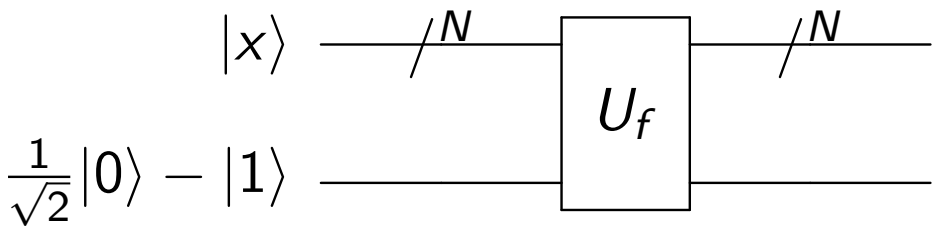

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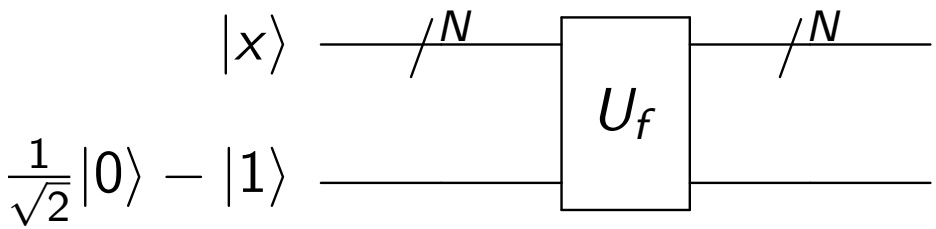

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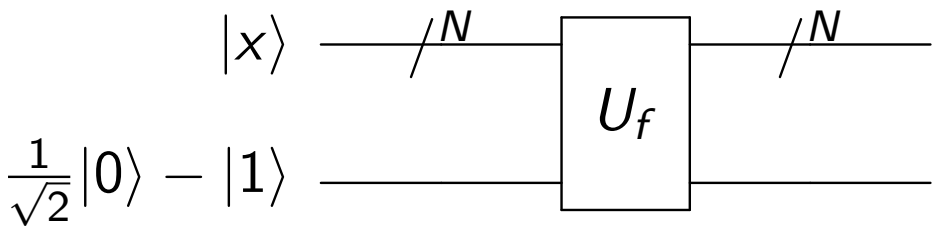

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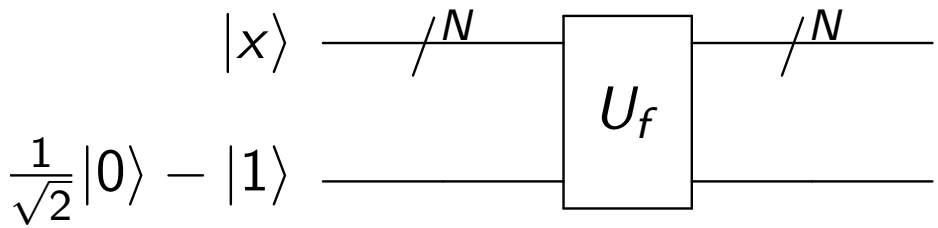

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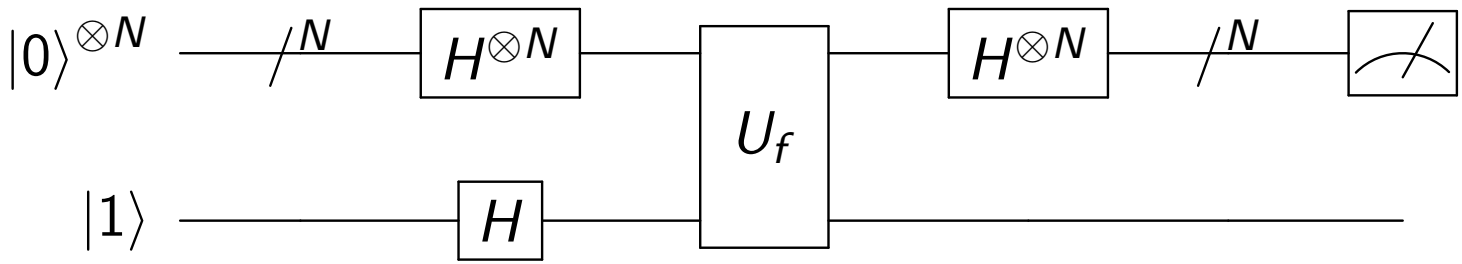
Deutsch-Josza algorithm



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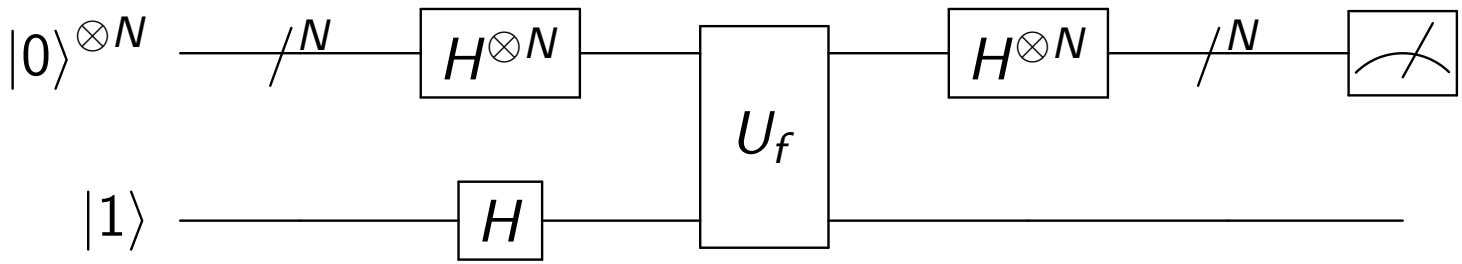

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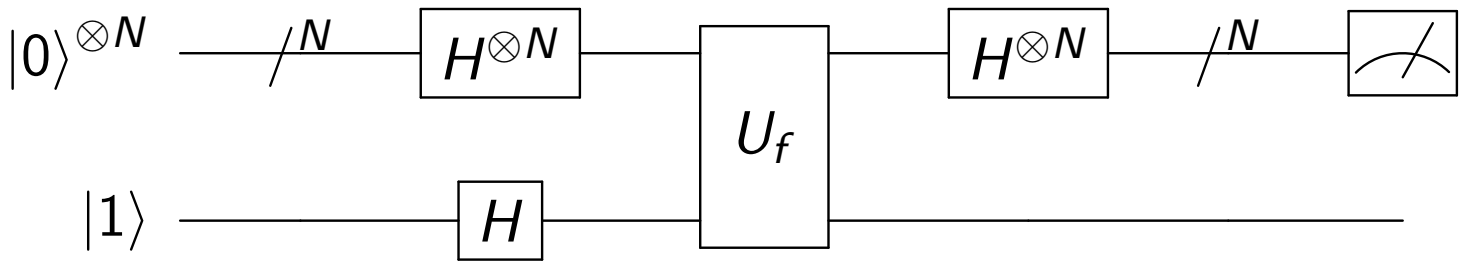

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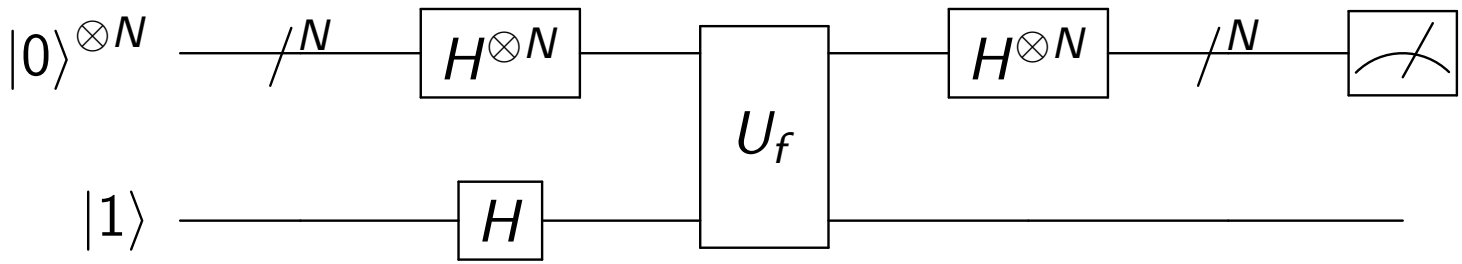

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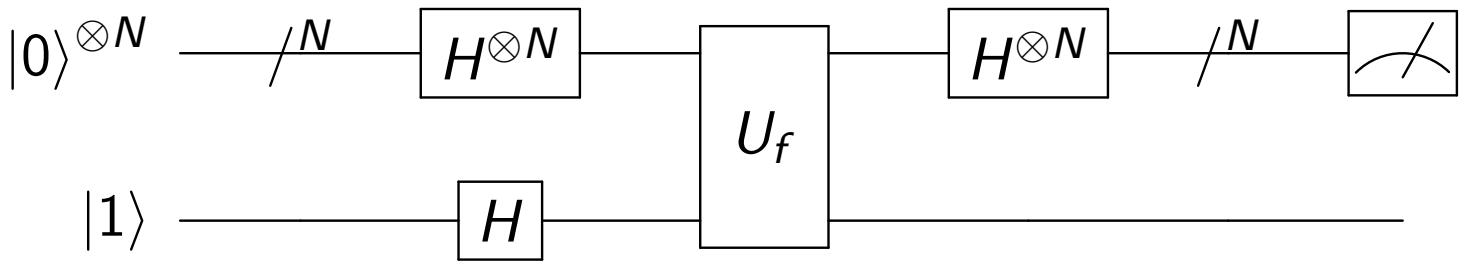

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- Exponential speed-up!

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One last thing before that...

A very useful relation

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You should remember this, it comes up a lot :)

Useful resources and references

- **A graphical quantum simulator** - http://www.quantumplayground.net/
- **Python SymPy for Quantum Computation** - http://digitalcommons.calpoly.edu/cgi/viewcontent.cgi?article=1038&context=physsp
- **Quantum Mechanics modules in SymPy** - http://docs.sympy.org/latest/modules/physics/quantum/index.html
- **Quantum Programming Languages** - http://www.quantiki.org/wiki/Quantum_Programming_Language
- **List of quantum algorithms** - http://math.nist.gov/quantum/zoo/
- **Lecture on quantum algorithms** - https://www.cs.berkeley.edu/~vazirani/algorithms/chap10.pdf
- **Quantum registers and algorithms** - http://people.cs.umass.edu/~strubell/doc/quantum_tutorial.pdf