Quantum Computation and Cryptography
Day 2

Alexandru Gheorghiu

The University of Edinburgh

Part II
Measurement, collapse and all that

Postulate IV - observation

The properties that can be observed about a quantum system are called observables. They correspond to Hermitian operators. The observed values are the eigenvalues of these operators, and the system "collapses" to the corresponding eigenstate. The probabilities of observation and collapse are given by Born's rule.

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Let's talk about **measurement** and **collapse**!

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Collapse $\leftrightarrow$ Projection

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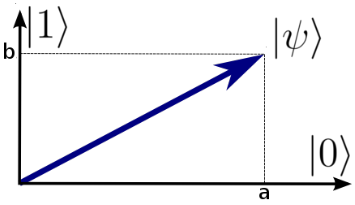

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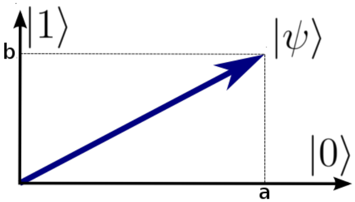

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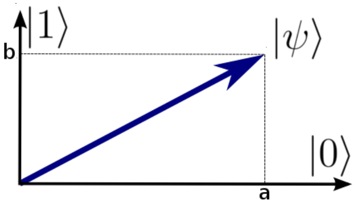

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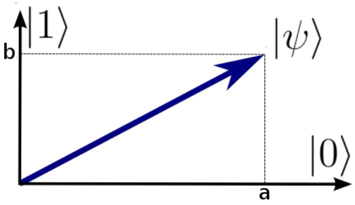

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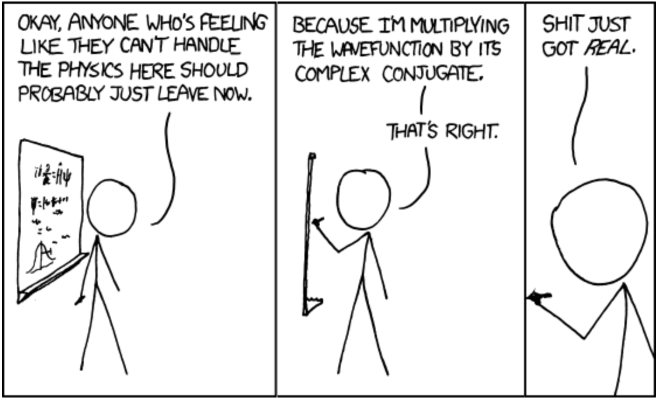
Always remember, when computing probabilities
probability $=$ *number* $\times$ *complex conjugate of number*



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And if we get outcome i, the state is projected to: $\frac{P_i|\psi\rangle}{\sqrt{p(i)}}$

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Let's look at some of its properties and relate to postulate IV

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$$Z|0\rangle = |0\rangle \quad Z|1\rangle = -|1\rangle$$

What are observables?

$$O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

First note that O is Z, from before

What are the eigenvalues of Z?
$+1$, -1

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Postulate IV in action:
Measuring the Z observable $\leftrightarrow$ measuring in the $|0\rangle$, $|1\rangle$ basis and the outcomes for each projection are $+1$ and -1

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I'm omitting some details, for now! :)

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State collapses to either $|00\rangle$ or $|11\rangle$ with equal probability!

(but we only acted on the first qubit)

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This is the expectation value of O (on the state $|\psi\rangle$)!


Achievement unlocked
Knows axioms of quantum mechanics

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A few more interesting things to say...

Discussion about measurement

Measurement is weird...

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The measurement problem

Many-worlds interpretation of quantum mechanics

Elitzur-Vaidman bomb detector

Back to the qubit

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So...

The Bloch sphere

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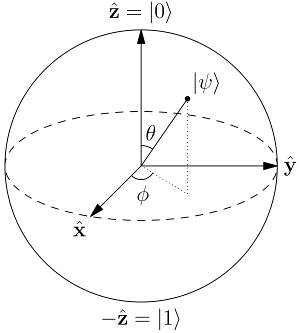

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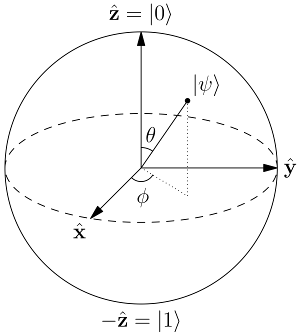

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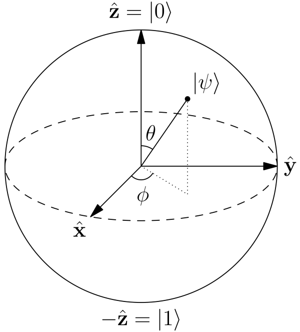

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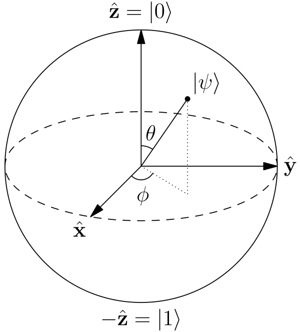

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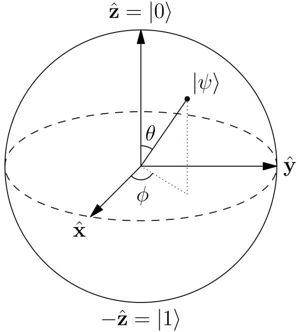

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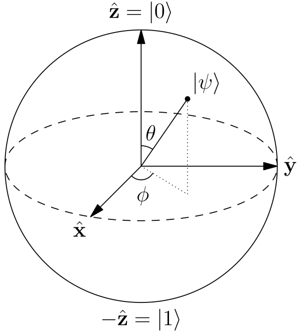

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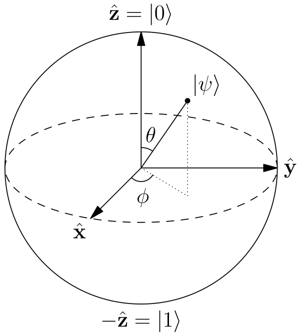

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But not in "real" space :)

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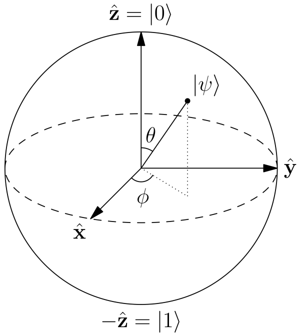

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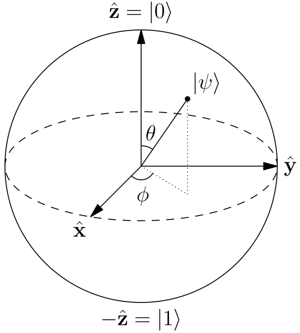

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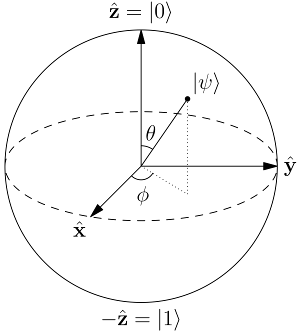
- Another useful one is T (aka the $\pi/8$ gate)



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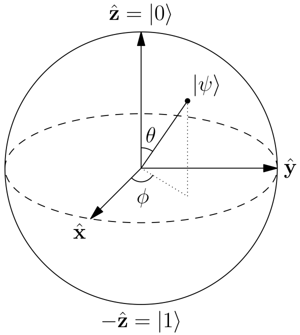

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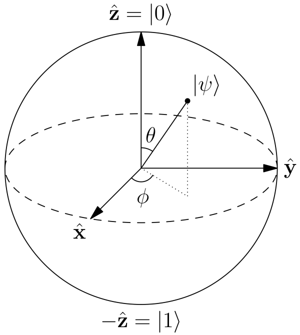

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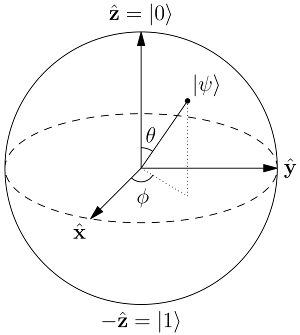

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  But set of all unitaries is <u>uncountable</u>!

Useful resources and references

- **The measurement problem** - https://en.wikipedia.org/wiki/Measurement_problem
- **Elitzur-Vaidman bomb detector** - https://arxiv.org/pdf/hep-th/9305002v2.pdf
- **Counterfactual computation** - http://rspa.royalsocietypublishing.org/content/457/2009/1175
- **Quantum Zeno effect** - https://en.wikipedia.org/wiki/Quantum_Zeno_effect

Useful resources and references

- Image on slide 9 (xkcd comic) - https://xkcd.com/849/
- Image on slide 19 (achievement unlocked) adapted from - https://uncommongeek.files.wordpress.com/2014/08/achievement-unlocked-template.jpg
- Image on slide 23 (Bloch sphere) - https://upload.wikimedia.org/wikipedia/commons/thumb/f/f4/Bloch_Sphere.svg/2000px-Bloch_Sphere.svg.png