Quantum Computation and Cryptography
Day 2

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Part I
The basics of quantum mechanics

"*There's no royal road to geometry*" - Euclid

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Quantum mechanics $=$ framework for building physical theories

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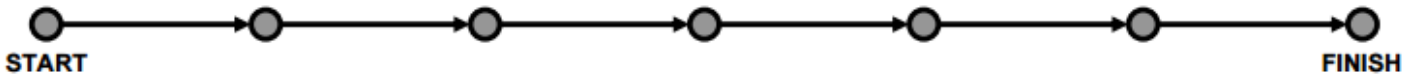
Quantum mechanics $=$ framework for building physical theories

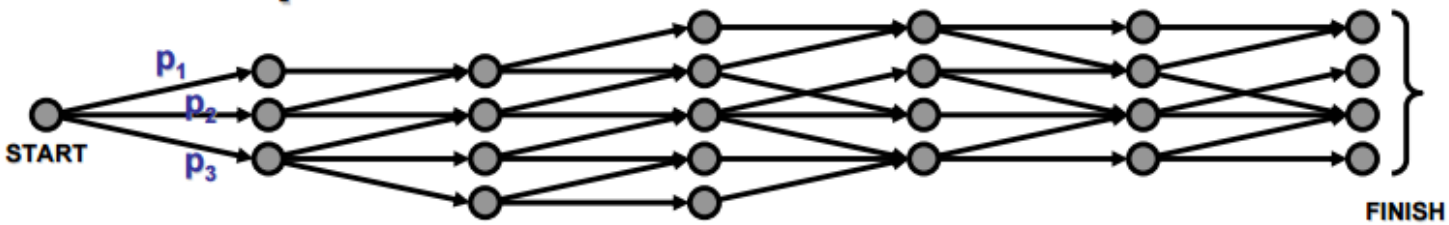
"*Quantum mechanics is actually remarkably simple, once you take all the physics out of it!*" - Scott Aaronson

Recap

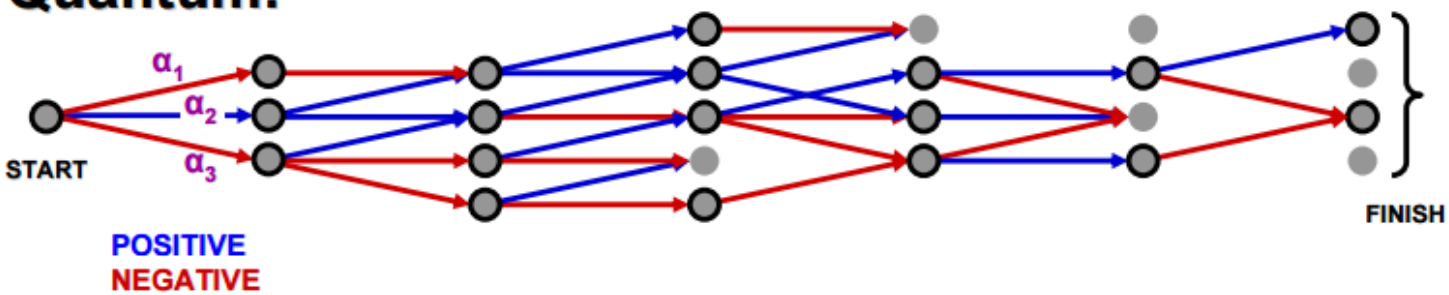
Classical deterministic:



Classical probabilistic:



Quantum:



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This is the essence of quantum mechanics!

Postulate I - state representation (intuition)

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The state of a system should be a norm 1 vector in a complex vector space

Postulate I - state representation

The state of an isolated quantum system is a ray of norm 1 in a complex vector space called Hilbert space.

Once upon a time in high-school

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$$e^{i\phi} = cos(\phi) + i\ sin(\phi)$$

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$$e^{i\phi} = cos(\phi) + i\ sin(\phi)$$

$$v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1 e^{i\phi_1} \\ r_2 e^{i\phi_2} \end{pmatrix} = e^{i\phi_1} \begin{pmatrix} r_1 \\ r_2 e^{i\Delta\phi} \end{pmatrix}$$

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We call ϕ_1 the **global phase** of v
And $\Delta\phi = \phi_2 - \phi_1$ the **relative phase**

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What's a Hilbert space? (denoted $\mathcal{H}$)

- Complex or real vector space with inner product $\langle , \rangle$
- $\forall\ x, y \in \mathcal{H},\ \langle x, y \rangle \in \mathbb{C}$
- $\forall\ x, y \in \mathcal{H},\ \langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\forall\ x_1, x_2, y \in \mathcal{H},\ \forall\ a, b \in \mathbb{C},$
  $\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$
- $\forall\ x, y_1, y_2 \in \mathcal{H},\ \forall\ a, b \in \mathbb{C},$
  $\langle x, a y_1 + b y_2 \rangle = \bar{a} \langle x, y_1 \rangle + \bar{b} \langle x, y_2 \rangle$
- $\forall\ x \in \mathcal{H},\ \langle x, x \rangle \geq 0$ (define $\|x\|^2 = \langle x, x \rangle$)

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What's a Hilbert space? (denoted $\mathcal{H}$)

It's a vector space with nice properties like **linearity** and **positivity**!

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Representation 1

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But $M_f v_0$ and $M_f v_1$ are not!

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- $|\psi\rangle = a\,|0\rangle + b\,|1\rangle \neq v = \begin{pmatrix} a \\ b \end{pmatrix}$
- Rather, $|\psi\rangle \leftrightarrow v$. They represent the same thing but are not equal

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$$|\text{electron path}\rangle = a_1\,|path_1\rangle + a_2\,|path_2\rangle + ...$$

Superposition

Oddly enough, politicians excel at quantum mechanics.

Postulate II - composing systems
Intuition

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If we have a 2^n dimensional vector and a 2^m dimensional vector
how do we get to the 2^{m+n} one?

Postulate II - composing systems

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Quantum states of a composite system live in a Hilbert space which is the tensor product of the Hilbert spaces of its components.

Postulate II

What is tensor product?

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An example will help us understand...

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Arbitrary vector

$$|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$
$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

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These are called **entangled** states, rest are **separable** states

Entanglement

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No! States are not isolated

Postulate III - dynamics
Intuition

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Why unitary evolution?

Linear and it preserves the square norm

Postulate III - dynamics

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The evolution of an isolated (closed) quantum system is governed by unitary operators.

Postulate III

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- We'll see next week, postulate III is equivalent to **Schrödinger's equation**

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

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- These groups have tremendous importance in physics as *symmetry* groups

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Single qubit examples

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$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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In general

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$$Z\,|\psi\rangle = aZ\,|0\rangle + bZ\,|1\rangle = a\,|0\rangle - b\,|1\rangle$$

$$|\chi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

$$CNOT\,|\chi\rangle =$$
$$aCNOT\,|00\rangle + bCNOT\,|01\rangle + cCNOT\,|10\rangle + dCNOT\,|11\rangle$$

Postulate III
Acting on general states

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

$$H\,|\psi\rangle = aH\,|0\rangle + bH\,|1\rangle = \frac{a}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{b}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H\,|\psi\rangle = \frac{a+b}{\sqrt{2}}\,|0\rangle + \frac{a-b}{\sqrt{2}}\,|1\rangle$$

$$X\,|\psi\rangle = aX\,|0\rangle + bX\,|1\rangle = a\,|1\rangle + b\,|0\rangle$$

$$Z\,|\psi\rangle = aZ\,|0\rangle + bZ\,|1\rangle = a\,|0\rangle - b\,|1\rangle$$

$$|\chi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

$$CNOT\,|\chi\rangle =$$
$$aCNOT\,|00\rangle + bCNOT\,|01\rangle + cCNOT\,|10\rangle + dCNOT\,|11\rangle$$

$$CNOT\,|\chi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|11\rangle + d\,|10\rangle$$

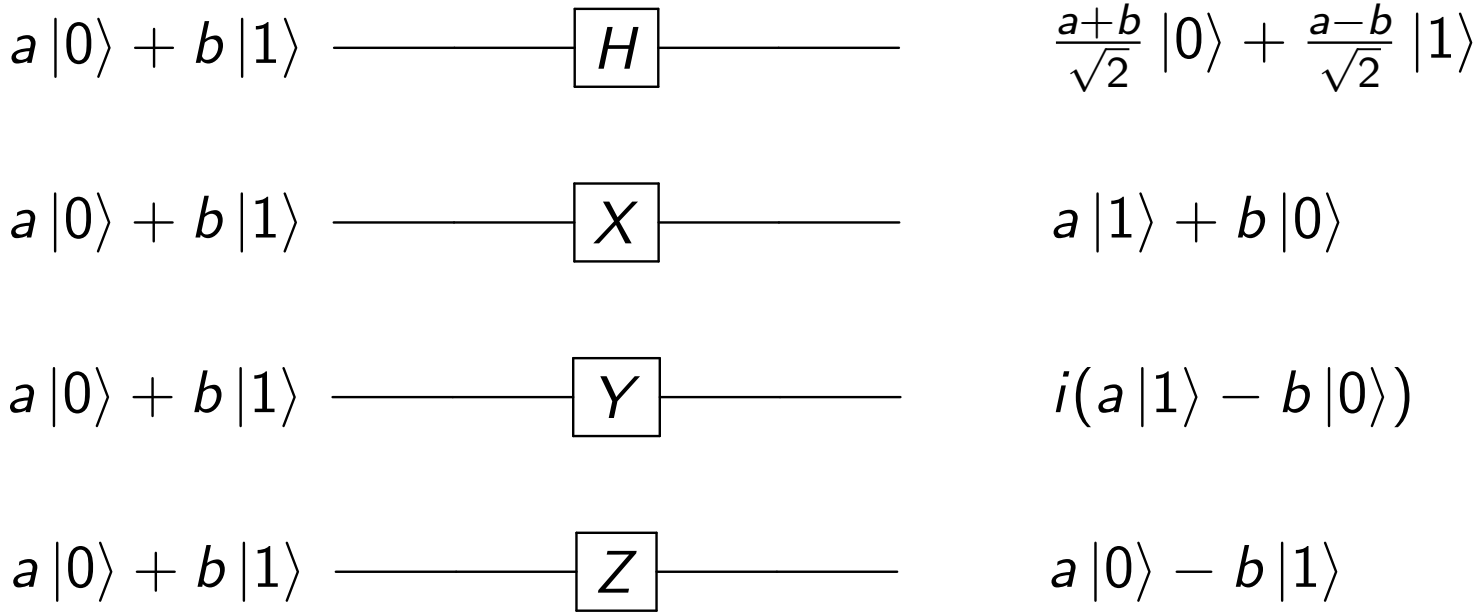
The circuit picture

The circuit picture

Everything we just mentioned can be represented as circuits

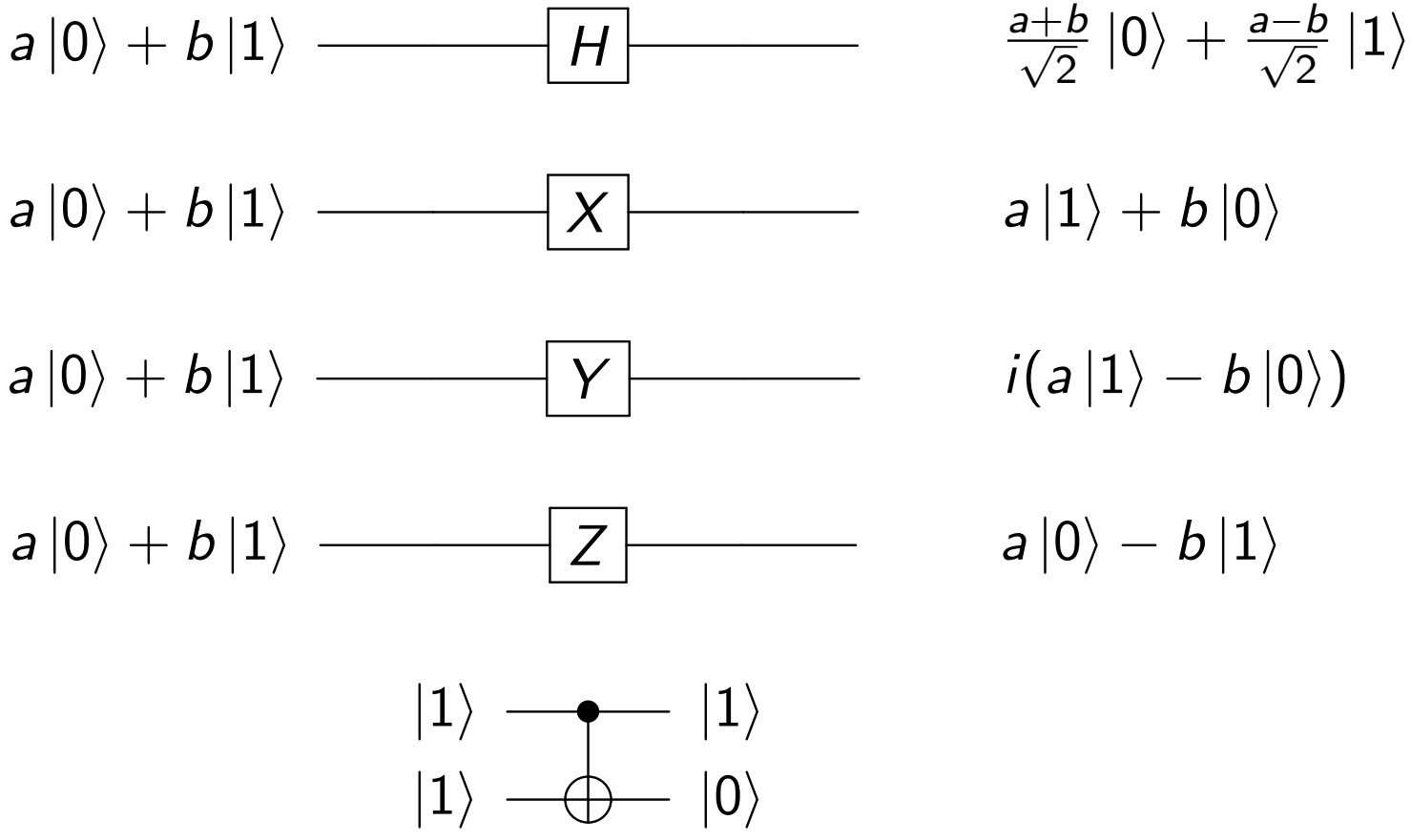
The circuit picture

Everything we just mentioned can be represented as circuits



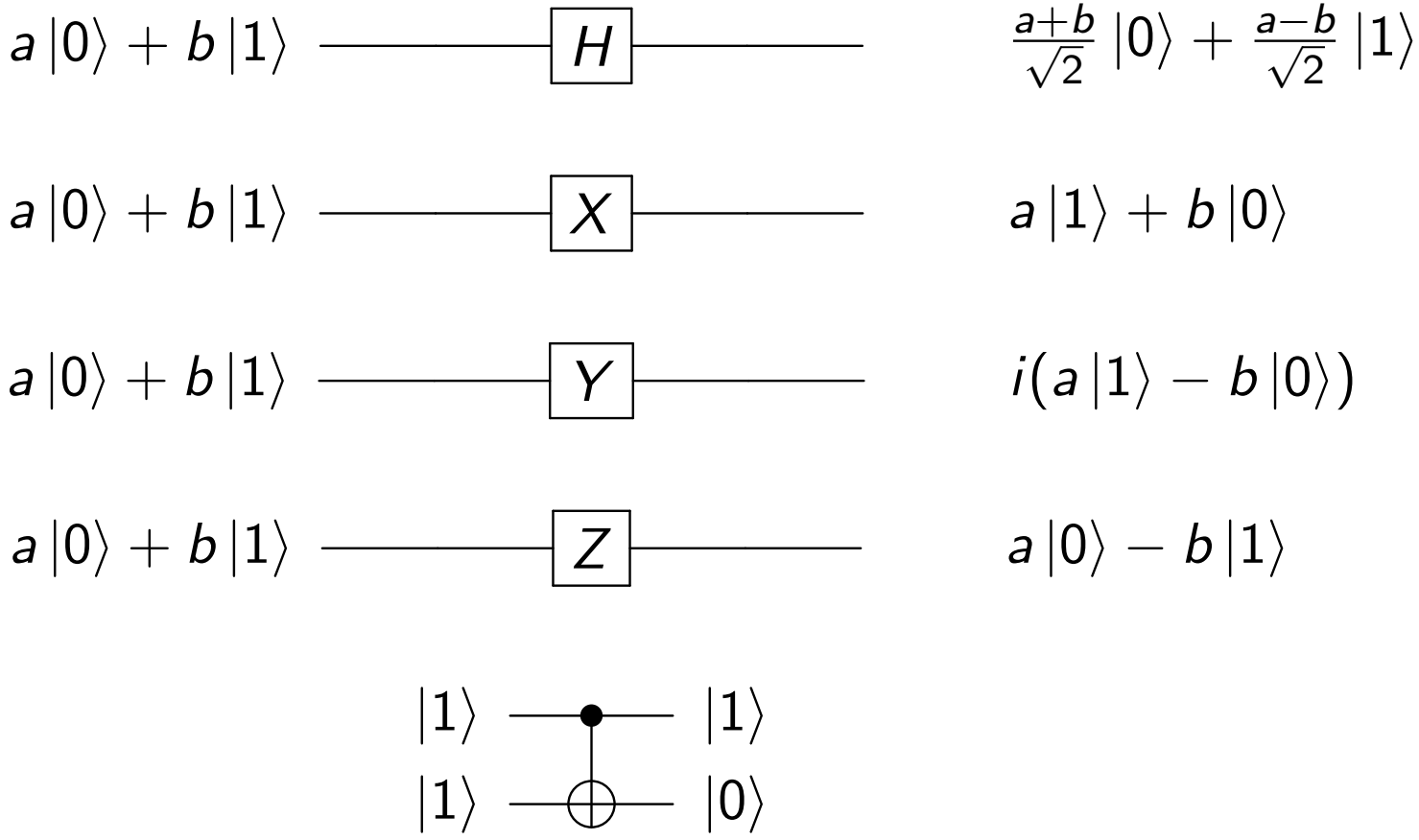
The circuit picture

Everything we just mentioned can be represented as circuits



The circuit picture

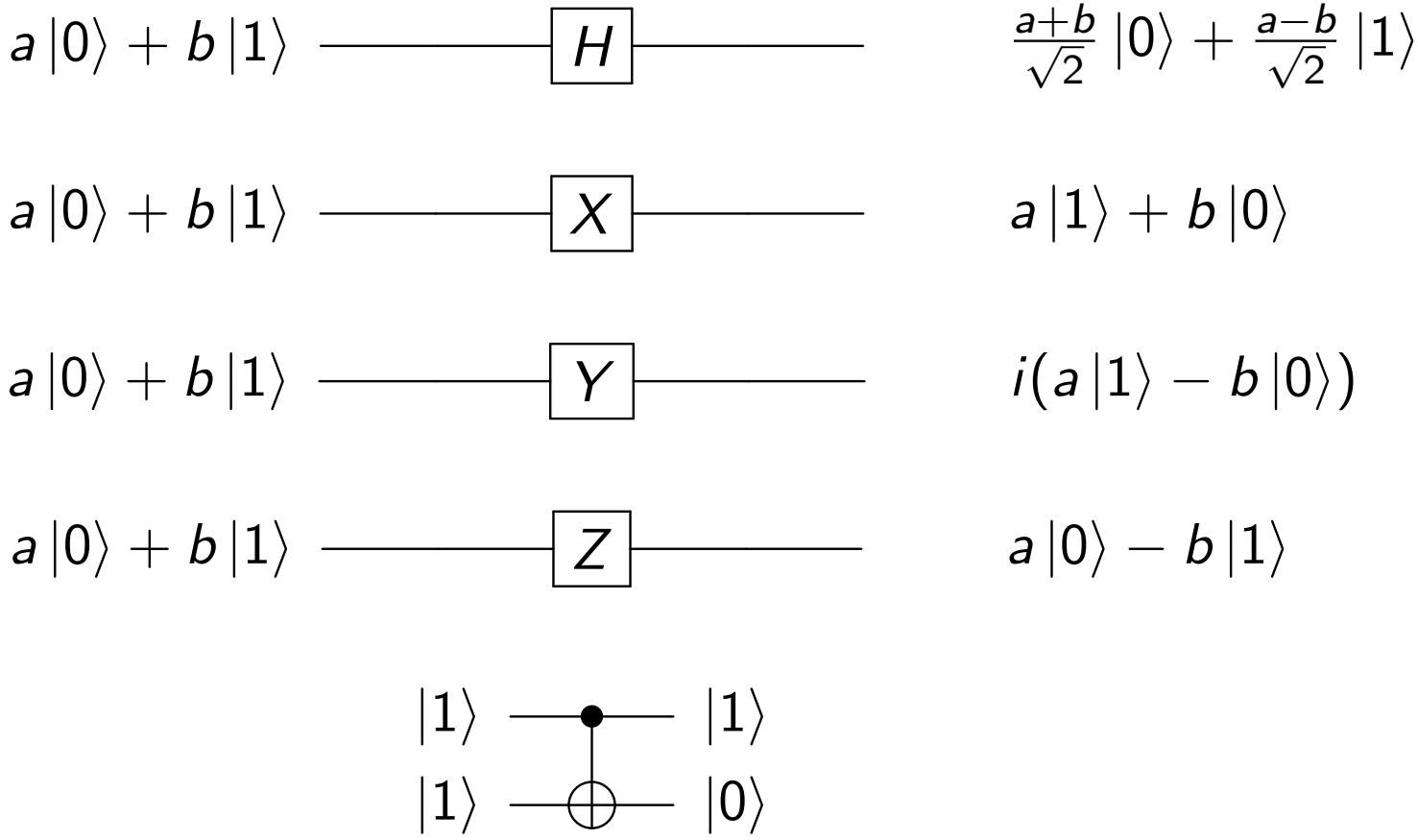
Everything we just mentioned can be represented as circuits



Note that all these gates/operators are their own inverses

The circuit picture

Everything we just mentioned can be represented as circuits



Note that all these gates/operators are their own inverses

So the pictures are valid in both directions!

The circuit picture

What is the output for the following circuit?

The circuit picture

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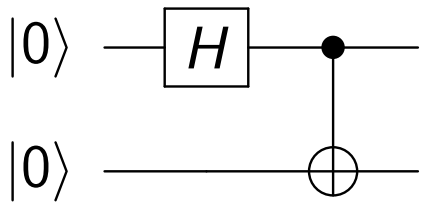
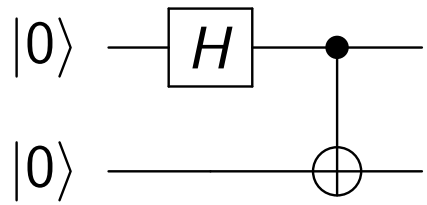


The circuit picture

What is the output for the following circuit?

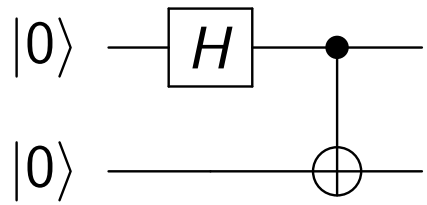


After first layer, we have:

$(H \otimes I)\,|00\rangle =$

The circuit picture

What is the output for the following circuit?

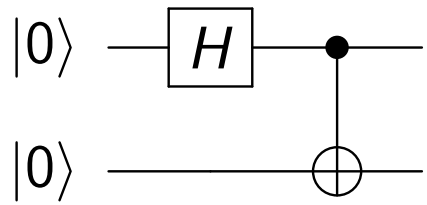


After first layer, we have:

$$(H \otimes I)\,|00\rangle = (H \otimes I)(|0\rangle \otimes |0\rangle) =$$

The circuit picture

What is the output for the following circuit?

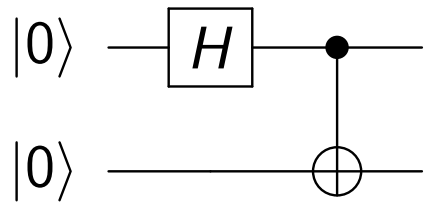


After first layer, we have:

$$(H \otimes I)\,|00\rangle = (H \otimes I)(|0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

The circuit picture

What is the output for the following circuit?



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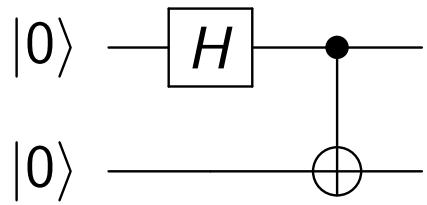
After second layer, we have:

$$CNOT\ \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) =$$

The circuit picture

What is the output for the following circuit?



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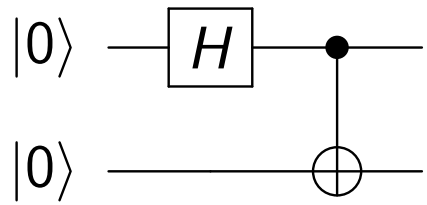
After second layer, we have:

$$CNOT\ \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(CNOT\,|00\rangle + CNOT\,|10\rangle)$$

The circuit picture

What is the output for the following circuit?



$$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

After first layer, we have:

$$(H \otimes I)\,|00\rangle = (H \otimes I)(|0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

After second layer, we have:

$$CNOT\ \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(CNOT\,|00\rangle + CNOT\,|10\rangle)$$

So we end up with:

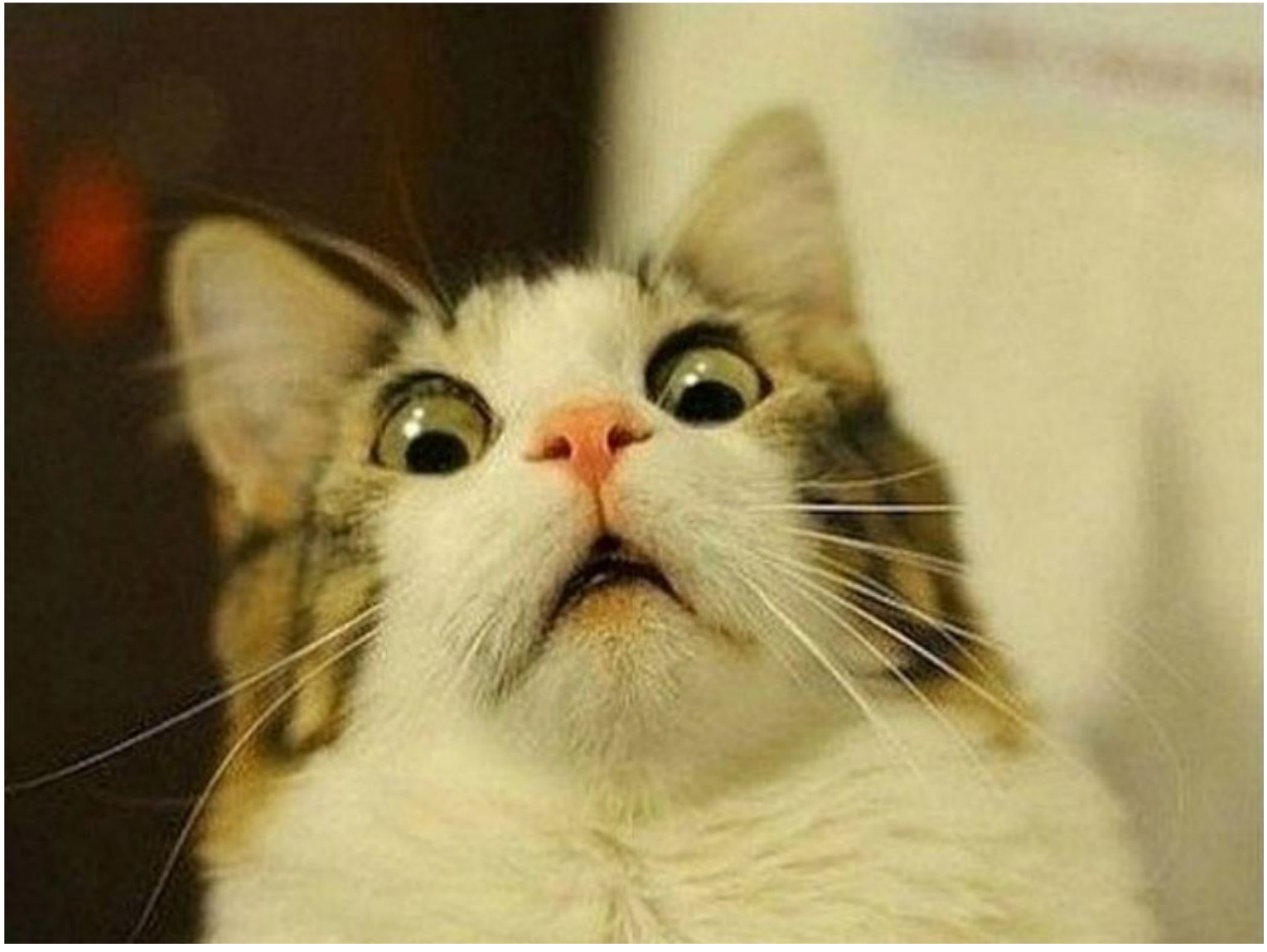
$$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

Postulate IV - observation

Brace yourselves...

Postulate IV - observation

The properties that can be observed about a quantum system are called observables. They correspond to Hermitian operators. The observed values are the eigenvalues of these operators, and the system "collapses" to the corresponding eigenstate. The probabilities of observation and collapse are given by Born's rule.



To be explained in part 2! :)

Useful resources and references

- **A nice quantum computing lecture** - https://qcintro.wordpress.com/lectures/
- **A nice introduction to quantum computing** - http://arxiv.org/pdf/0708.0261v1.pdf
- **Quantum computing for non-physicists** - http://arxiv.org/pdf/quant-ph/9809016v2.pdf
- **A more mathematical lecture** - http://www.theory.caltech.edu/~preskill/ph219/
- **Quantum theory from five reasonable axioms** - http://arxiv.org/pdf/quant-ph/0101012v4.pdf
- **Density matrix formalism** - http://www.cithep.caltech.edu/~fcp/physics/quantumMechanics/densityMatrix/densityMatrix.pdf

Useful resources and references

- **Postulates of QM** - any quantum mechanics textbook. Recommended: Introduction to Quantum Mechanics, http://www.amazon.com/Introduction-Quantum-Mechanics-2nd-Edition/dp/0131118927
- **Postulates applied to computing and anything about quantum information** - Quantum Computation and Quantum Information, http://www.amazon.com/Quantum-Computation-Information-Anniversary-Edition/dp/1107002176
- Image on slide 4 (comparison) - http://download2.cerimes.fr/canalu/documents/fuscia/quantum.turing.test_13249/kashefi.pdf
- Image on slide 13 (SMBC comic) - http://www.smbc-comics.com/comic/2011-04-21
- Image on slide 34 (cat reaction) - http://dailynewsdig.com/wp-content/uploads/2013/06/20-Funny-Shocked-Cat-Memes-3.jpg